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Multiplying matrices 2

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In this second leaflet on matrix multiplication we delve more deeply into the conditions under which matrices can be multiplied together and look further at the sizes of the matrices which can result from matrix multiplication. We will go on to look at a very useful property of the identity matrix.

Recall that if A has size $p \times q$, that is, it has p rows and q columns, and B has size $r \times s$, that is, it has r rows and s columns, we can only multiply them together if $q = r$. When this is so, the result of multiplying them together, C say, is a $p \times s$ matrix.

$$\begin{array}{ccc} A \quad B & = & C \\ p \times \underbrace{q \quad r}_{q=r} \times s & & p \times s \end{array}$$

Example. Find the product MN when $M = \begin{pmatrix} 3 & 2 & 1 \\ 4 & -3 & 2 \\ 5 & 4 & 3 \end{pmatrix}$ and $N = \begin{pmatrix} 1 & -2 \\ -2 & 3 \\ 3 & -4 \end{pmatrix}$.

Solution. The first matrix has size 3×3. The second matrix has size 3×2. Clearly the number of columns in the first is the same as the number of rows in the second. The multiplication can be performed and the result will be a 3×2 matrix.

$$MN = \begin{pmatrix} 3 & 2 & 1 \\ 4 & -3 & 2 \\ 5 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -2 & 3 \\ 3 & -4 \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ 16 & -25 \\ 6 & -10 \end{pmatrix}$$

Now let us see what happens if we try and multiply these matrices together the opposite way around. That is, we try to find NM:

$$NM = \begin{pmatrix} 1 & -2 \\ -2 & 3 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} 3 & 2 & 1 \\ 4 & -3 & 2 \\ 5 & 4 & 3 \end{pmatrix}$$

The first matrix has size 3×2 and the second has size 3×3. In this order, the number of columns in the first (2) is NOT the same as the number of rows in the second (3). This means that it is not possible to perform this calculation.

> For any two matrices A and B, even when the product AB is defined it may be the case that, because of the size of the matrices, BA is not defined.

Example. For the matrices $C = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $D = \begin{pmatrix} 3 & -7 \\ 5 & 1 \\ -1 & 2 \end{pmatrix}$ find, if possible, CD and DC.

Solution. C has size 2×3 and D has size 3×2. So when we consider CD, the first matrix has three columns and the second has three rows. So, it is possible to evaluate CD and the result will be a 2×2

matrix:

$$CD = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 3 & -7 \\ 5 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1\times 3 + 2\times 5 + 3\times(-1) & 1\times(-7) + 2\times 1 + 3\times 2 \\ 4\times 3 + 5\times 5 + 6\times(-1) & 4\times(-7) + 5\times 1 + 6\times 2 \end{pmatrix} = \begin{pmatrix} 10 & 1 \\ 31 & -11 \end{pmatrix}$$

Now we consider DC: the first matrix is 3×2 and the second is 2×3. So, it is possible to evaluate DC and the result will be a 3×3 matrix:

$$\begin{aligned} DC &= \begin{pmatrix} 3 & -7 \\ 5 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 3\times 1 + (-7)\times 4 & 3\times 2 + (-7)\times 5 & 3\times 3 + (-7)\times 6 \\ 5\times 1 + 1\times 4 & 5\times 2 + 1\times 5 & 5\times 3 + 1\times 6 \\ (-1)\times 1 + 2\times 4 & (-1)\times 2 + 2\times 5 & (-1)\times 3 + 2\times 6 \end{pmatrix} \\ &= \begin{pmatrix} -25 & -29 & -33 \\ 9 & 15 & 21 \\ 7 & 8 & 9 \end{pmatrix} \end{aligned}$$

So, we see that both CD and DC exist but they are not the same.

For any two matrices C and D, even if the products CD and DC are defined, in general CD and DC will have different sizes. In general, $CD \neq DC$ - matrix multiplication is **not commutative**.

Even when the products CD and DC have the same size, these products may well not be equal. Consider, for example, if $C = \begin{pmatrix} 2 & 4 \\ 5 & 3 \end{pmatrix}$ and $D = \begin{pmatrix} 3 & 6 \\ -1 & 9 \end{pmatrix}$,

$$CD = \begin{pmatrix} 2 & 4 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 3 & 6 \\ -1 & 9 \end{pmatrix} = \begin{pmatrix} 2 & 48 \\ 12 & 57 \end{pmatrix} \qquad \text{but} \qquad DC = \begin{pmatrix} 3 & 6 \\ -1 & 9 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 36 & 30 \\ 43 & 23 \end{pmatrix}$$

Because matrix multiplication is not commutative it is always important to ensure that you write down the matrices to be multiplied in the order intended. When we write AB we say that B is **pre-multiplied** by A. Equivalently, A is **post-multiplied** by B.

Finally, we look at an important property of identity matrices: if we multiply any matrix by an identity matrix of the appropriate size the result is the same as the matrix we started with. For example, verify that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -4 & 7 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -4 & 7 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 8 \\ -11 \\ 5 \end{pmatrix} = \begin{pmatrix} 8 \\ -11 \\ 5 \end{pmatrix}$$

This result will be important when we come on to using matrices for solving simultaneous equations. Likewise, if we post-multiply by an identity matrix the same property holds. Verify that

$$\begin{pmatrix} 7 & 8 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 8 \end{pmatrix}$$

and

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

Further leaflets in this series explain other matrix operations.