

First Order Ordinary Differential Equation

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Introduction

Prerequisites: You will need to know about trigonometry, differentiation, integration, complex numbers in order to make the most of this teach-yourself resource.

We are looking at equations involving a function $y(x)$ and its first derivative:

$$\frac{dy}{dx} + P(x)y = Q(x) \quad (1)$$

We want to find $y(x)$, either explicitly if possible, or otherwise implicitly.

Direct Integration

This method is used to solve ODEs in the form:

$$\frac{dy}{dx} = f(x)$$

These ODEs can be solved as follows:

$$\begin{aligned} dy &= f(x)dx \\ \text{so: } y &= \int f(x)dx \end{aligned}$$

Example:

$$\begin{aligned} \frac{dy}{dx} &= 3x^2 - 6x + 5 \\ y &= x^3 - 3x^2 + 5x + C \text{ with } C \text{ constant of integration} \end{aligned}$$

The constant of integration can be anything, unless you have boundary conditions. In the previous example, if you have $y(0)=0$ then:

$$y(0) = C = 0$$



Separation of Variables

This method is used to solve ODEs in the form:

$$\frac{dy}{dx} = f(x)g(y)$$

These ODEs can be solved as follows:

$$\frac{dy}{g(y)} = f(x)dx$$

$$\int \frac{dy}{g(y)} = \int f(x)dx$$

Example:

$$\begin{aligned}\frac{dy}{dx} &= \frac{2x}{y+1} \\ (y+1)dy &= 2xdx \\ \int (y+1)dy &= \int 2xdx \\ \frac{1}{2}y^2 + y &= x^2 + C \text{ with } C \text{ constant of integration}\end{aligned}$$

Homogeneous Equations

A first order homogeneous differential equation is a differential equation in the form:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

These ODEs can be solved by making the substitution $y(x) = v(x) \cdot x$ where v is a function of x . Then we have:

$$\text{Using the product rule: } \frac{dy}{dx} = \frac{dv}{dx} \cdot x + v$$

$$\text{Inserting in the ODE: } \frac{dv}{dx} \cdot x + v = f(v)$$

$$\text{Re-arranging: } \frac{dv}{f(v) - v} = \frac{dx}{x}$$

$$\int \frac{dv}{f(v) - v} = \int \frac{dx}{x}$$



Example

$$\frac{dy}{dx} = \frac{x + 3y}{2x}$$

Rearranging, we get:

$$\frac{dy}{dx} = \frac{1}{2} + \frac{3y}{2x}$$

$$x \frac{dy}{dx} + v = \frac{1}{2} + \frac{3}{2}v$$

$$x \frac{dv}{dx} = \frac{1}{2} + \frac{1}{2}v$$

$$\frac{dv}{1+v} = \frac{dx}{2x}$$

$$\ln(1+v) = \frac{1}{2} \ln(x) + C = \ln(x^{1/2}) + C \text{ with } C \text{ constant of integration}$$

$$\text{Taking the log on both sides: } v = e^C x^{1/2} - 1$$

$$\text{with } y = v \cdot x, \quad y = kx^{3/2} - x, \text{ with } k = e^C$$

Integrating Factor

This method is used to solve ODEs in the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

These ODEs can be solved as follows: multiply both sides of the equation by **the integrating factor:** $e^{\int P(x)dx}$. Then:

$$e^{\int P(x)dx} \frac{dy}{dx} + P(x)e^{\int P(x)dx} y = Q(x)e^{\int P(x)dx}$$

Now we see that, using the product rule and the chain rule:

$$e^{\int P(x)dx} \frac{dy}{dx} + P(x)e^{\int P(x)dx} y = \frac{d}{dx} \left(e^{\int P(x)dx} y \right)$$

Therefore:

$$\begin{aligned} \frac{d}{dx} \left(e^{\int P(x)dx} y \right) &= Q(x)e^{\int P(x)dx} \\ y &= e^{-\int P(x)dx} \int Q(x)e^{\int P(x)dx} dx \end{aligned}$$

Example

$$\frac{dy}{dx} - y = x \text{ with } P(x) = -1 \text{ and } Q(x) = x$$

The integrating factor is $e^{\int -dx} = e^{-x}$ and:

$$\begin{aligned} e^{-x} \frac{dy}{dx} - e^{-x} y &= x e^{-x} \\ \frac{d}{dx} (y e^{-x}) &= x e^{-x} \end{aligned}$$

Using integration by part: $y e^{-x} = -e^{-x}(1+x) + C$ with C constant of integration

$$y = -(1+x) + C e^x$$



Bernoulli Equations

Bernoulli equations are of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

These equations are solved with the following method:

- Multiply both sides of the equation by y^{-n} , the equation becomes:

$$y^{-n} \frac{dy}{dx} + P(x)y^{1-n} = Q(x)$$

- Make the change of variable $v = y^{1-n}$, then:

$$\frac{dv}{dx} = (1-n)y^{-n} \frac{dy}{dx}$$

$$\text{so } \frac{1}{1-n} \frac{dv}{dx} + P(x)v = Q(x)$$

The latest form of the equation can be solved for v using the method of the integrating factor. Finally, y can be found from the relationship $y = v^{n-1}$.

Example:

$$\frac{dy}{dx} + \frac{1}{x}y = xy^2$$

$$y^{-2} \frac{dy}{dx} + \frac{1}{x}y^{-1} = x$$

$$v = y^{-1}, \quad \frac{dv}{dx} = -\frac{1}{y^2} \frac{dy}{dx}$$

$$\frac{dv}{dx} - \frac{v}{x} = -x$$

$$\text{The integrating factor is IF} = e^{\int -\frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x}$$

$$\frac{1}{x} \frac{dv}{dx} - \frac{1}{x^2} v = -1$$

$$\frac{1}{x} v = \int -dx$$

$$v = -x^2 + Cx$$

$$y = \frac{1}{-x^2 + Cx}$$



Exercises

$$(a) x \frac{dy}{dx} = 5x^3 + 4$$

$$(b) x(y-3) \frac{dy}{dx} = 4y$$

$$(c) (2y-x) \frac{dy}{dx} = 2x+y \text{ with } y(2) = 3$$

$$(d) x \frac{dy}{dx} - y = x^3 + 3x^2 - 2x$$

$$(e) (1+x^2) \frac{dy}{dx} + 3xy = 5x \text{ with } y(1) = 2$$

$$(f) 2 \frac{dy}{dx} + y = y^3(x-1)$$

Answers

$$(a) y = \frac{5}{3}x^3 + 4 \ln x + C$$

$$(d) y = \frac{1}{2}x^3 + 3x^2 - 2x \ln x + Cx$$

$$(b) y - 3 \ln y = 4 \ln x + C$$

$$(e) y = \frac{5}{3} + \frac{\sqrt{8}}{3}(1+x^2)^{-3/2}$$

$$(c) \left(\frac{y}{x}\right)^2 - \frac{y}{x} - 1 + x^{-2} = 0$$

$$(f) y = (x + Ce^{-x})^{-1/2}$$

