Eigenvalues and eigenvectors

mccp-croft-0901 September 9, 2010

Introduction

This leaflet summarises how eigenvalues and eigenvectors of a square matrix are found.

The characteristic equation

Given a square $n \times n$ matrix A, we can form a new matrix $A - \lambda I$, where λ is an (as yet) unknown number and I is the $n \times n$ identity matrix. For example, if we start with the 2×2 matrix

$$A = \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}$$

then we can form

$$A - \lambda I = \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which is simplified to

$$A - \lambda I = \begin{pmatrix} 3-\lambda & 1 \\ -1 & 5-\lambda \end{pmatrix}.$$

If we now evaluate the determinant of $A - \lambda I$ we obtain what is called the **characteristic polynomial** of A. In this case,

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 \\ -1 & 5-\lambda \end{vmatrix} = (3-\lambda)(5-\lambda) - (1)(-1) = \lambda^2 - 8\lambda + 16.$$

So the characteristic polynomial in this example is the quadratic polynomial $\lambda^2 - 8\lambda + 16$. The **characteristic equation** is

$$\lambda^2 - 8\lambda + 16 = 0.$$

In the case of a 3×3 matrix the characteristic polynomial will be cubic, and the algebra gets a little more tedious, but the method of calculation is the same.

Eigenvalues

The eigenvalues of a matrix A are the solutions of its characteristic equation. For example the eigenvalues of $A = \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}$ are found by solving $\lambda^2 - 8\lambda + 16 = 0$. Thus

$$\begin{aligned} \lambda^2 - 8\lambda + 16 &= 0 \\ (\lambda - 4)(\lambda - 4) &= 0 \\ \lambda &= 4 \qquad \text{(twice)}. \end{aligned}$$

In this example there is one (repeated) eigenvalue, $\lambda = 4$. You should note that in a more general 2×2 case, the solution of the quadratic characteristic equation may yield two real distinct eigenvalues, or perhaps two complex eigenvalues.

© Author: Tony Croft
Loughborough University

Reviewer: Leslie Fletcher
Liverpool John Moores University

Eigenvectors

Given an $n \times n$ matrix A, and having found its eigenvalues, its eigenvectors are found as follows: For *each* eigenvalue separately, you need to solve the system of simultaneous equations

$$A\mathbf{x} = \lambda\mathbf{x}$$

where $\mathbf{x}$ is a column vector of size n. For example, in the 2×2 and 3×3 cases we have, respectively

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{and} \qquad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

You will find that the simultaneous equations so formed always have an infinite number of solutions for each eigenvalue. Each of the solutions, $\mathbf{x}$, is called an **eigenvector** of A.

To find the eigenvectors of $A = \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}$, corresponding to eigenvalue $\lambda = 4$, we solve

$$\begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 4 \begin{pmatrix} x \\ y \end{pmatrix}$$

that is, by simplification,

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Writing out these equations explicitly, we obtain

$$-x + y = 0.$$

The solution is $x = t$, $y = t$ for any value of t. The variable t is called a **free variable** and it can take any value. Hence there is an infinite number of solutions. Some of these are

$$x = 1, y = 1; \qquad x = -3, y = -3; \qquad x = \frac{1}{2}, y = \frac{1}{2}.$$

Each of these solutions provides an eigenvector of A corresponding to eigenvalue $\lambda = 4$. Note that they are all scalar multiples of each other and we usually quote just one, and write, for example, that the eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Exercises

1. Calculate the eigenvalues and corresponding eigenvectors of $A = \begin{pmatrix} 5 & 6 \\ 2 & 1 \end{pmatrix}$.

Answers

1. $\lambda = -1, 7$. For $\lambda = -1$, the eigenvector is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. For $\lambda = 7$, the eigenvector is $\begin{pmatrix} 1 \\ \frac{1}{3} \end{pmatrix}$.

© Author: Tony Croft
Loughborough University

Reviewer: Leslie Fletcher
Liverpool John Moores University