Solving Differential Equations by Separating Variables

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Introduction

Suppose we have the first order differential equation

$$P(y)\frac{dy}{dx} = Q(x)$$

where $Q(x)$ and $P(y)$ are functions involving x and y only respectively. For example

$$y^2\frac{dy}{dx} = \frac{1}{x^3} \qquad \text{or} \qquad \frac{1}{y^2}\frac{dy}{dx} = \frac{x-3}{x^3}.$$

We can solve these differential equations using the technique of **separating variables**.

General Solution

By taking the original differential equation

$$P(y)\frac{dy}{dx} = Q(x)$$

we can solve this by separating the equation into two parts. We move all of the equation involving the y variable to one side and all of the equation involving the x variable to the other side, then we can integrate both sides. Although $\frac{dy}{dx}$ is not a fraction, we can intuitively treat it like one to move the "dx" to the right hand side. So

$$P(y)\frac{dy}{dx} = Q(x) \Leftrightarrow \int P(y)\,dy = \int Q(x)\,dx.$$

Example

Let us find the general solution of the differential equation

$$y^2\frac{dy}{dx} = \frac{1}{x^3}.$$

$$\begin{aligned}
y^2\frac{dy}{dx} = \frac{1}{x^3} &\Leftrightarrow \int y^2\,dy = \int \frac{1}{x^3}\,dx \\
&\Leftrightarrow \int y^2\,dy = \int x^{-3}\,dx \\
&\Leftrightarrow \frac{y^3}{3} = \frac{-x^{-2}}{2} + c \quad \text{where } c \text{ is a constant} \\
&\Leftrightarrow y^3 = \frac{-3}{2x^2} + 3c \\
&\Leftrightarrow y = \sqrt[3]{\frac{-3}{2x^2} + 3c}
\end{aligned}$$

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Example

To find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{y^2(x-3)}{x^3}$$

we first need to move the y^2 to the left hand side of the equation. Then we move the dx to the right hand side of the equation and integrate both sides.

$$\begin{aligned}
\frac{dy}{dx} = \frac{y^2(x-3)}{x^3} &\Leftrightarrow \frac{1}{y^2}\frac{dy}{dx} = \frac{x-3}{x^3} \\
&\Leftrightarrow \int \frac{1}{y^2}\,dy = \int \frac{x-3}{x^3}\,dx \\
&\Leftrightarrow \int \frac{1}{y^2}\,dy = \int \frac{x}{x^3} - \frac{3}{x^3}\,dx \\
&\Leftrightarrow \int y^{-2}\,dy = \int \frac{1}{x^2} - \frac{3}{x^3}\,dx \\
&\Leftrightarrow \int y^{-2}\,dy = \int x^{-2} - 3x^{-3}\,dx \\
&\Leftrightarrow -y^{-1} = -x^{-1} + \frac{3x^{-2}}{2} + c \quad \text{where } c \text{ is a constant} \\
&\Leftrightarrow \frac{-1}{y} = -\frac{1}{x} + \frac{3}{2x^2} + c \\
&\Leftrightarrow \frac{1}{y} = \frac{1}{x} - \frac{3}{2x^2} - c \\
&\Leftrightarrow \frac{1}{y} = \frac{2x}{2x^2} - \frac{3}{2x^2} - \frac{2cx^2}{2x^2} \\
&\Leftrightarrow \frac{1}{y} = \frac{2x-3-2cx^2}{2x^2} \\
&\Leftrightarrow y = \frac{2x^2}{2x-3-2cx^2}
\end{aligned}$$

Exercises

Find the general solution of

1. $\dfrac{dy}{dx} = y(1+e^x)$ 2. $\dfrac{dy}{dx} = \dfrac{x}{y}$ 3. $\dfrac{dy}{dx} = 9x^2y$ 4. $\dfrac{4}{y^3}\dfrac{dy}{dx} = \dfrac{1}{x}$

Answers

1. $y = e^{x+e^x+c}$ 2. $y = \pm\sqrt{x^2+2c}$ 3. $y = e^{3x^3+c}$ 4. $y = \pm\sqrt{\dfrac{-2}{\ln|x|+c}}$

Note that the $\pm$ symbol like in Exercise 2 means that the differential equation has two sets of solutions, $y = \sqrt{x^2+2c}$ and $y = -\sqrt{x^2+2c}$.

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