Teaching Problem-solving in Undergraduate Mathematics

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> “Ill-used persons, who are forced to load their minds with a score of subjects against an examination, who have too much on their hands to indulge themselves in thinking or investigation, who devour premiss and conclusion together with indiscriminate greediness, who hold whole sciences on faith, and commit demonstrations to memory, and who too often, as might be expected, when their period of education is passed, throw up all they have learned in disgust, having gained nothing really by their anxious labours, except perhaps the habit of application.
>
> Cardinal John Henry Newman

> “The object of mathematical rigour is to sanction and legitimate the conquests of intuition, and there never was any other object for it.
>
> Jacques Hadamard

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Chapter 1

Introduction

Our purpose in this Guide is to argue the case for putting problem-solving at the heart of a mathematics degree; for giving students a flavour, according to their capabilities, of what it is to be a mathematician; a taste for rising to a mathematical challenge and overcoming it. Our purpose is also to make it easier for colleagues who share our vision to find ways of realising it in their own teaching.

The Guide properly begins in Chapter 2, where we define our terms and discuss the views of education theorists on the role of problem-solving in mathematics teaching. Next comes John Mason's critique of Pólya's work from a modern viewpoint, and this is followed by Bob Burn's account of his experience of writing a problem-solving course from scratch. In Chapter 5 we draw on the experience of colleagues, and, more particularly, on our six case studies, to offer practical advice on ways of introducing serious problem-solving into the curriculum. Sue Pope, in Chapter 6, considers the role of computers in aiding students' problem-solving. Finally, in Chapter 7, we present the details of our six case-studies of modules where problem-solving has been taught as part of a mathematics degree programme in a U.K. university. *Readers more interested in the practicalities of starting their own problem-solving modules may like to read the case studies first and then go straight to Chapter 5*.

The ability to solve previously unseen problems, independently and with confidence, is an important skill for a graduating mathematician. The Q.A.A.[1] Benchmark (2007) for M.S.O.R.[2] recognises this fact, mentioning the practice of problem-solving 16 times, including in the following context:

> “Employers greatly value the intellectual ability and rigour and the skills in reasoning that these learners will have acquired, their familiarity with numerical and symbolic thinking, and the analytic approach to problem-solving that is their hallmark.

[1]Quality Assurance Agency

[2]Mathematics, Statistics and Operational Research

More recently the HE Mathematics Curriculum Summit, held in January 2011 at the University of Birmingham, included in its final report the following (Rowlett, 2011, p. 19):

> Problem-solving is the most useful skill a student can take with them when they leave university. It is problematic to allow students to graduate with first class degrees who cannot handle unfamiliar problems.

The report concludes with 14 recommendations for developing higher education teaching, the first three of which relate to problem-solving: sharing good practice, crafting sequences of suitable problems, and pooling a collection of wider teaching resources.

Problem-solving therefore is widely recognised for its importance, but the ways in which it may be taught, and indeed what 'problem-solving' means, remain elusive. Chapter 2 explains what *we* mean by problem-solving, what to us makes a 'good problem' and how problem-solving relates to mathematical thinking. It also reviews the history of teaching problem-solving and the various theories that have been applied to the pedagogy. In Chapter 5 we discuss ways we believe are effective in developing students' problem-solving skills. Those interested simply in starting a problem-solving module of their own, or in introducing problem-solving in existing modules, could skip forward to this chapter which contains practical advice we believe will be useful to begin such a task. We hope, however, they will take some time to review the theoretical and historical aspects of their proposed activity.

The three main authors would like to extend their thanks to Bob Burn, John Mason, and Sue Pope, for their valuable contributions to this guide. Furthermore, we are most grateful to our case-study departments, and interviewees in particular, for the time they have devoted to helping us document their problem-solving practices. Our final thanks go to the Maths, Stats and OR Network, and the National HE STEM Programme, for the funds and support that, without which, this project would not have existed.

Chapter 2

Background to Problem-solving in Undergraduate Mathematics

Matthew Badger, Trevor Hawkes and Chris Sangwin

What does it mean to be a mathematician, and what is the purpose of a mathematics degree? Any answer to the second question follows, in part, from that to the first: a mathematics degree is the first stage in a mathematical apprenticeship. Mathematicians form a community of practice around mathematical activity, and so we must begin by considering this. The importance of problem-solving to mathematics was summed up by the Hungarian mathematician Paul Halmos in his article *The Heart of Mathematics* (1980, p. 519):

> “What does mathematics really consist of? Axioms (such as the parallel postulate)? Theorems (such as the fundamental theorem of algebra)? Proofs (such as Gödel’s proof of undecidability)? Definitions (such as the Menger definition of dimension)? Theories (such as category theory)? Formulas (such as Cauchy’s integral formula)? Methods (such as the method of successive approximations)?
>
> Mathematics could surely not exist without these ingredients; they are all essential. It is nevertheless a tenable point of view that none of them is at the heart of the subject, that the mathematician’s main reason for existence is to solve problems, and that, therefore, what mathematics really consists of is problems and solutions.

A professional mathematician engaged in research considers problems that no-one has yet solved, and therefore to give students a reasonable apprenticeship in mathematics, to give them an experience of what it means to *do mathematics*, we need to put them in unfamiliar situations with problems which, to them, are novel with the expectation that they must seriously tackle them for themselves.

In this chapter we define what we mean by ‘problem’, argue for the place of problem-solving in an undergraduate mathematics curriculum, and consider the ways in which problem-solving has been taught to undergraduates in the past. We err on the side of the working mathematician tasked with teaching problem-solving to undergraduate students at university. Problem solving in school, where pressure from externally prescribed compulsory high-stakes examinations for the whole cohort of students interfere, may well be quite a different issue. There are also other interesting perspectives which are more theoretical, for example those that consider issues such as cognition and social dynamics. As a starting point for these we recommend Mason et al. (2010) and Schoenfeld (1994).

2.1 Exercises and Problems

We perceive a commonly held view in the mathematics community that everyone understands what a mathematical problem really is and can recognise one when they see one. For example, the reports we quoted in the introduction stress the importance of problem-solving in the mathematics curriculum but do not feel the need to explain what they mean by it. The tacit understanding is that the readers already know and agree on what it means. However, this widespread belief in a consensus does not bear closer scrutiny, as John Mason argues at length in Section 3.1.

For our subsequent discussion we make the following distinctions:

- A mathematical *question* is a task that can be assigned to a student, who is expected to carry it out and, at some later stage, to submit an account of their answer. Thus it is a very broad, generic term used in all kinds of assessment.
- A mathematical *exercise* is a question whose solution involves only routine procedures. Having learnt the relevant techniques, a student will be able to follow an obvious strategy and systematically apply the techniques in sequence of easy steps to reach a correct solution.

Here is an example of an exercise:

▼ Example 2.1.1

Calculate

$$\int_{-\pi}^{\pi} \sin(2x)\ \mathrm{d}x.$$

To answer this question, students need to know the indefinite integral of $\sin(nx)$ and the value of $\cos x$ at -2π and 2π. For students who know how to apply this knowledge, the question is an exercise. Alternatively, students who recognise that sine is an odd function and know how to cancel positive and negative areas can solve the question without integration. For these students too it is an exercise.

Although we are championing problem-solving here, we also believe that exercises are a valuable, indeed indispensable, part of direct instruction in mathematics. The teacher demonstrates a particular skill—in the case of Example 2.1.1, using integration between limits—and the students attempt to learn the skill by analogy, following the teacher's example and applying the same procedure to closely-related questions. This process often happens in two stages: in lectures, the teacher demonstrates a skill; in examples classes, students attempt to recreate the teacher's method, sometimes with the encouragement of a tutor. Lectures can be an effective and low-cost way of transmitting information to a large number of students (Bligh, 1998, p. 11), and most U.K. mathematics departments use this lecture/tutorial format to teach most of their modules.

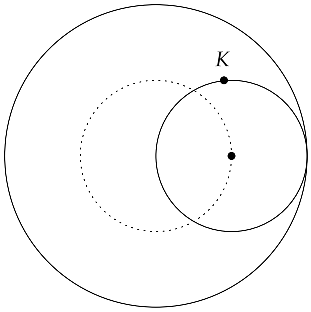
2.1.1 Problems

For our purposes a *problem* is a question that is not an exercise. To put it another way, *a problem is a question whose process for answering it is unclear*. Notice that we cannot classify a question on its own as a problem or exercise: it is as much a function of the *particular student* as it is the mathematical processes which lead to a correct solution. This definition provides an immediate practical difficulty: one person's problem is another's exercise. The teacher's exercise may well be the student's problem. Historically, Example 2.1.1 was a mathematical research problem, see Edwards (1979). Whenever we discuss differences between exercises and problems, we can only ever do so in terms of individual students at a particular point in their teaching. One may reasonably assume, however, that given a particular cohort of students on an undergraduate degree programme, the division of questions into exercises and problems will be similar for each student in the cohort.

The defining characteristic of problems as we speak of them is *novelty*, and in response to this novelty their answering requires *creativity*. Example 2.1.2 presents a question from geometry that, for most contemporary undergraduates, will be a *problem*. Few will easily relate this question to some previously encountered ideas or procedures, and so they cannot immediately think of how to go about answering it; to them it is *novel*.

▼ Example 2.1.2 — *Circles*

Let K be a point on a circle sitting inside and touching a stationary circle of twice its diameter. Describe the path of K as the smaller circle rolls round the larger one without sliding.



A relatively small deviation from the norm may be all that is needed to make a problem novel. Clarity in how to proceed depends on perception and recognition. In illustration of this, Example 2.1.3 is a famous question that, in spite of its apparent straightforwardness, initially poses real difficulties to many students:

▼ Example 2.1.3 — *Students and Professors*

Write an equation to express the statement "There are six times as many students as professors at this university". *Use S for the number of students and P for the number of professors.*

This problem was initially studied by Clement et al. (1981), who posed the question to 150 calculus students; 37% of whom answered incorrectly; two-thirds of incorrect answers being "$6S = P$". Since Clement et al's original paper, this particular problem has been revisited on occasion (Fisher, 1988; Stacey and MacGregor, 1993) and students continue to perform poorly when attempting it. In this instance the problem's novelty comes from requiring students to translate the words into mathematics – a problem of *interpretation*.

Another example of a classic problem on which much research has focused is given in Example 2.1.4, devised by Wason (1968). People perform poorly, again struggling with interpretation and frequently mistaking "*every card which has a D on one side has a 7 on the other*" with "*every card which has a D or a 7 on one side has a 7 or a D on the other*". An equivalent problem, presented in terms of social interactions—"*Every person drinking alcohol is 18 or older.*"—is answered correctly much more often (Griggs and Cox, 1982).

▼ Example 2.1.4 — *Wason Selection Test*

Imagine you have a four cards on a table and every card has a letter on one side and a number on the other.

With the cards placed on a table, you see

$\boxed{\text{D}}\quad\boxed{3}\quad\boxed{\text{K}}\quad\boxed{7}$

*Turn over the fewest cards to establish the truth of the following statement "*Every card which has a D on one side has a 7 on the other.*"*

Examples 2.1.3 and 2.1.4 are two problems whose statements are precise and unambiguous; nevertheless, students struggled with the correct interpretation. Another possible situation is a problem whose statement is open to interpretation, where the terms may not be properly defined or the logic is suspect. In this case the student's task could be to fix the statement so that its meaning is clear and its conclusion is true.

A celebrated discussion of this situation is given in Imre Lakatos's *Proofs and Refutations* (1976), a socratic dialogue in which a teacher and their students attempt to prove or disprove the statement "*For all polyhedra,* $V - E + F = 2$". In this case the students begin with differing and unclear interpretations of *polyhedron*, *vertex*, *edge*, and *face*. The extract below (Lakatos et al., 1976, p. 14) demonstrates one of the issues the students have to resolve: how to define the set of objects called 'polyhedra' to ensure that the conclusion is true? They must first decide whether to alter their interpretation of the terms in the conjecture, or the conjecture itself:

> “ DELTA: But why accept the counterexample? We proved our conjecture—now it is a theorem. I admit that it clashes with this so-called 'counterexample'. One of them has to give way. But why should the theorem give way, when it has been proved? It is the 'criticism' that should retreat. It is fake criticism. This pair of nested cubes is not a polyhedron at all. It is a monster, a pathological case, not a counterexample.
>
> GAMMA: Why not? A polyhedron is a solid whose surface consists of polygonal faces. And my counterexample is a solid bounded by polygonal faces.
>
> TEACHER: Let us call this definition *Def. 1*.

Through a sequence of proposed proofs and counterexamples, the class eventually reaches both a proof of the statement and rigorous definitions for the terms used in it.

Traditional direct instruction avoids situations such as this, precisely because fruitful definitions (e.g. continuity, convergence) are difficult to formulate.

▼ Example 2.1.5 — *Goldbach's conjecture*

Every even integer greater than 2 can be expressed as the sum of two primes.

The statement of Goldbach's conjecture—Example 2.1.5—can be understood by children at secondary school, and yet it has remained open for over two-and-a-half centuries. With this problem the novelty is one of *strategy*: where can one begin? What constitutes a solution? What arguments can be brought to bear? While it would clearly be unreasonable to expect undergraduate mathematics students to solve this, they can certainly be expected to attempt the following Example 2.1.6, taken from Durham University's problem-solving module (see Section 7.2), which illustrates a similar kind of novelty. For a typical beginning undergraduate its statement is very likely to be clear, and yet a way to approach it and the form of an acceptable solution are often not.

▼ Example 2.1.6 — *Square Differences*

Which numbers can be written as the difference of two perfect squares, e.g. $6^2 - 2^2 = 32$*?*

Another method for bringing novelty to a question is to introduce of new *concepts*, or redefine old ones. In the next Example 2.1.7, students have to work with a new definition of multiplication on a set of real numbers.

▼ Example 2.1.7 — *Group*

The set $\mathbb{R}_c$*, defined as* $\mathbb{R}\backslash c$*, for some* $c \in \mathbb{R}$*, is combined with the multiplication operation defined by*

$$x \odot y = (x - c)(y - c) + c.$$

Verify that $\mathbb{R}_c$ *is a group under this operation. Find the identity and inverse elements.*

For a student who is new to the subject, this question is likely to present an algebraic challenge; it is easy for the novice to confuse the two types of product . A more experienced student of group theory will probably find it a routine exercise to prove associativity and the existence of an identity and inverse elements.

▼ Example 2.1.8 — *Hamiltonian Graph*

A Hamiltonian path *is a path that visits each vertex exactly once. A graph is* Hamiltonian *if it has a Hamiltonian path. Prove that a complete graph on $n \geq 2$ vertices is Hamiltonian.*

In Example 2.1.8 the potential or intended novelty is the introduction of the concept of a *Hamiltonian graph*. In a traditional module, the implications of a newly-defined concept would most likely be fully explored during the lectures. This contrasts with the approach often taken in problem-based learning (see Section 2.6) where students are given new concepts without comment and are left to investigate their implications and understand their significance by working through problems about them. And as students gain experience and confidence in problem-solving, they can even be encouraged to create and try out their own novel definitions to see if they have fruitful consequences.

We have argued that novelty is a key feature in a mathematical problem and have illustrated three ways it can be introduced:

- novelty in a problem's formulation, calling for careful interpretation
- novelty in the kind of strategies that will lead to a solution
- novelty of the concepts the problem addresses

We have also stressed that novelty is crucially a function of the student's existing knowledge and previous experience.

By solving problems, students learn to apply their mathematical skills in new ways; they develop a deeper understanding of mathematical ideas and get a taste of the experience of 'being a mathematician'. With success comes a sense of achievement, even enjoyment, and possibly an appetite for more. Repeated practice under these circumstances increases their confidence, builds up their stamina, and rewards perseverance. By the process of regular reflection on their problem-solving activities, students can store up these experiences for future use.

In Section 2.2 we will take a closer look at the ways such experiences can benefit students. Before that, we end this section by considering two further types of questions closely related to what we mean by mathematical problems; *word problems* and *puzzles*.

2.1.2 Word Problems and Modelling

Word problems, which include *algebra story problems*, are a question presented to students in rhetorical form. The student's first task is to translate the text into more formal mathematics. This has already been illustrated by Example 2.1.3. In yet another of his influential works, *Mathematical Discovery* (1962), Pólya asserts that word problems in particular deserve a special place on the mathematical curriculum:

> "I hope that I shall shock a few people in asserting that the most important single task of mathematical instruction in the secondary schools is to teach the setting up of equations to solve word problems. Yet there is a strong argument in favor of this opinion. In solving a word problem by setting up equations, the student translates a real situation into mathematical terms; he has an opportunity to experience that mathematical concepts may be related to realities, but such relations must be carefully worked out. – Pólya, 1962, p. 59

Translating words into mathematics is still an important skill; when modelling a real-world situation, we also need to represent it mathematically. We then operate on the resulting mathematical abstractions, but periodically check for a meaningful correspondence between the real situation and symbolic abstraction. Unlike problem-solving, when modelling we are very often required to make *approximations* to enable the resulting equations to be solved. A good problem, and a good modelling question, both require decisions on the part of the solver. This is all part of *making sense*. Modelling is closely related to problem-solving, and engineering and science, particularly physics, make use of mathematics in this way. It is argued in Sangwin (2011) that word problems form the start of modelling.

Evidence has consistently shown that word problems are initially likely to be "problematic" until significant practice has been undertaken. Hence, these ultimately may become exercises and not genuine *problems* in the way that we define them. However, many of our students find these particularly difficult and they are often encountered as problem-solving tasks.

Pure mathematics is less concerned with "real world" problems than with the intellectual patterns of consequences and connections. However, word problems are a particular genre of their own in elementary mathematics with a long history (Gerofsky, 2004). Algebra story problems are found in both Egyptian and Mesopotamian mathematics from before 2000BC (Gerofsky, 2004; Robson, 2008; Høyrup, 1990). Example 2.1.9 has reoccurred perennially since Alcuin of York's *Problems to Sharpen the Young*, written around 775, see Hadley and Singmaster (1992).

▼ **Example 2.1.9 — *Dog and Hare***

A dog starts in pursuit of a hare at a distance of thirty of his own leaps from her. If he covers as much ground in two leaps as she in three, in how many of his leaps will the hare be caught?

A version of this problem is thought to be (Swetz, 1972) in 九章算術 (*The Nine Chapters on the Mathematical Art*), a book compiled in the first century AD from texts dated between 1000BC and 200BC. Because such questions are a part of mathematical culture and history, we suggest they also retain a place of modest importance in pure mathematics, and as part of problem-solving.

2.1.3 Puzzles and Recreational Mathematics

It is useful to distinguish another subclass of problems called *puzzles*. A puzzle is a particular type of problem that usually requires little prior mathematical knowledge. There have been attempts to define puzzles, for instance as tasks requiring 'lateral thinking' (De Bono, 1971). Michalewicz and Michalewicz (2008) suggest that the nature of an (educational) puzzle is encapsulated by four criteria: generality (explaining some universal mathematical problem-solving principle), simplicity, a 'Eureka' factor and an entertainment factor. To us, a puzzle is a question that is free-standing (requiring little or no prior knowledge), lightweight (lacking gravitas and without ramifications), and needing ingenuity to solve it. Frequently, such ingenuity will be in the form of a trick or sudden insight, providing the Eureka factor mentioned by Michalewicz and Michalewicz (2008).

A few examples will show what we mean by a puzzle – the first has a laborious solution involving an infinite series, and an instant solution when 'the penny drops' (the 'Eureka!' moment).

▼ **Example 2.1.10 — *Bee and trains***

Two model trains are travelling toward each other on the same track, each at a speed 5 *km h*$^{-1}$ *along the track. When they are* 50 *metres apart, a bee sets off at* 10 *km h*$^{-1}$ *from the front of one train, heading toward the other. If the bee reverses its direction every time it meets one of the trains, how far will it have travelled before it must fly upwards to avoid a grisly demise?*

A mathematical puzzle need not have wider impact on mathematical knowledge, or be part of a circle of ideas; it can be happily self-contained. Sometimes, however, there are useful principles involved, or techniques that can be stored up for future use. The next puzzle shows the value of extending a diagram by adding extra information.

▼ **Example 2.1.11 — *Diagonal***

Two line segments are drawn as diagonals on the sides of a cube so that they meet at a vertex of the cube. What is the angle between the segments?

The trick is to draw a third diagonal to produce an equilateral triangle. Another momentary pleasant surprise.

It is tempting to dismiss puzzles for being educationally lightweight, but we argue this would be shortsighted—puzzles are valuable in a variety of ways. They can stimulate students' interest and provoke curiosity without requiring special preparation or background knowledge; they can suggest methods that are useful in other contexts; and they give students a sense of achievement, even aesthetic pleasure, when they find a satisfying solution. In a pilot project at the University of Birmingham, first-year Chemical Engineering students took part in workshops using Puzzle-based Learning, a teaching method proposed by Michalewicz and Michalewicz (2008).

The project's main aim was to develop new learning resources to enable STEM staff to incorporate puzzle-based learning in their teaching. By working consistently on puzzles in a supportive environment, students learn to accept intellectual challenges, adopt novel and creative approaches, develop modelling and estimation skills, practice recognition of cases, and show tenacity. Above all, they learn to take responsibility for their learning.

2.2 The Value of Problem-solving

Mathematics is a systematic way of structuring thought and arguments, which is tied closely to a coherent body of associated knowledge. Mathematical *practice* is concerned with finding solutions to problems, whether related to practical problems (applied) or internal to mathematics (pure). Indeed, mathematical research is itself a form of problem-solving. Problem-solving is therefore an important, if not *the most* important, component of our discipline.

In undertaking problem-solving a student needs to develop both intellectual and temperamental qualities. Students need to:

- Identify essential steps and work out a strategy.

- Seek out relevant knowledge and bring it to bear.
- Use structured and logical arguments.
- Carry through a plan accurately using a sequence of linked steps.
- Know when to turn back in a dead end and try a different tack.
- Organise, present, and defend their solution.
- Submit a solution to the scrutiny of the teacher or their peers.
- Explore the consequences of their solution, ask further questions, experiment with hypotheses and conclusions, try out generalisations.

Some temperamental qualities needed for problem-solving require a student to:

- Accept the challenge and take responsibility for finding a solution.
- Be tenacious.
- Be prepared to take risks.
- Tolerate frustration.
- Look back at their own work in a critical way.
- Accept criticism from others.

As with most human endeavours, effortful practice under suitable encouragement and guidance builds confidence, and breeds success.

Employers value mathematicians because of their ability to solve problems; they expect our students to be able to do these things in unfamiliar situations. Very few students need to use the theorems of group theory in later life; to them the value of learning group theory, beyond its intrinsic beauty and interest, is how a they change because they have made the effort to do so.

If we value something, we need to teach it. Students on our courses pick up messages about what we value by how we behave. Unlike school work, which follows an externally prescribed curriculum, 'university mathematics' is *precisely what we define it to be*. If, as an extreme and artificial example, the programme consists only of showing students methods for solving particular kinds of exercises then mathematics itself is seen as this activity. If, on the other hand, students are expected to tackle unfamiliar problems and to puzzle things out for themselves, this activity also becomes part of mathematics. Consider the contrapositive of the opening sentence of this paragraph: if we don't teach something, then we don't value it. Let us parody this. Are we really *not going to teach* problem-solving?

Like any large body of knowledge, mathematics, as it grows, gets systematised and its subject matter put into different compartments. This is a sensible and necessary process, however as a consequence students tend to think of mathematics in terms of areas such as number theory, calculus and algebra. Mathematics was not always so tidy, and it is intriguing and instructive to look at the ad-hoc methods used historically to solve quadrature problems and compare these with the modern equivalents which benefit from algebra and calculus (see Edwards, 1979). Because of this systematising instinct, it is tempting to forget the importance of grappling with unseen problems, and of developing strategies and emotional resources to do so. Just as athletes need to develop stamina, so to do our students when attempting to solve problems. Many students currently at U.K. universities undertaking mathematics degree courses have been very successful at school, as measured by the public examinations system. To them, many of the tasks they have been asked to do so far were not problems. It often comes as a shock to realise that some mathematical problems might require many hours of thought, and even that people might enjoy the challenge of tackling them. Again, we return to ask what is mathematical activity to these students, and to us as researchers and teachers?

It has long been known that students must struggle to solve problems independently and construct their own meaning. Mathematics education is the art of helping students to reinvent the wheel.

For example, as early as 1543 in one of the first English textbooks on arithmetic, Robert Recorde acknowledges this as follows.

> “ **Master.** *So may you have marked what I have taught you. But because thys thynge (as all other) must be learned [surely] by often practice, I wil propounde here ii examples to you, whiche if you often do practice, you shall be rype and perfect to subtract any other summe lightly ...*
> **Scholar.** *Sir, I thanke you, but I thynke I might the better doo it, if you did showe me the woorkinge of it.*
> **Master.** *Yea but you muste prove yourselfe to do som thynges that you were never taught, or els you shall not be able to doo any more then you were taught, and were rather to learne by rote (as they cal it) than by reason.* – Recorde, 1543, Sig.F, i, v[1]

This fundamental tension between *telling students the correct method* to solve a problem and *requiring them to solve for themselves* is particularly marked in mathematics.

The following, satirical, criticism of strict instruction is a reminder to us that dissatisfaction with education is nothing new.

> “ I was at the mathematical school, where the master taught his pupils after a method scarce imaginable to us in Europe. The proposition and demonstration, were fairly written on a thin wafer, with ink composed of a cephalic tincture. This, the student was to swallow upon a fasting stomach, and for three days following, eat nothing but bread and water. As the wafer digested, the tincture mounted to his brain, bearing the proposition along with it. But the success has not hitherto been answerable, partly by some error in the *quantum* or composition, and partly by the perverseness of lads, to whom this bolus is so nauseous, that they generally steal aside, and discharge it upwards, before it can operate; neither have they been yet persuaded to use so long an abstinence, as the prescription requires. – Swift, 1726, Chapter 4

There are serious practical difficulties of *how* to teach problem-solving, which we seek to address in the subsequent sections and by providing case-studies from colleagues' attempts to address this issue.

2.3 Students' Previous experience with Problem-solving

The American educational psychologist, David Ausubel, gave the following advice to teachers:

> “ The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly – Ausubel, 1968, p. vi

Ausubel's dictum certainly applies to teaching students to solve problems. If a student has never grappled before with a substantial mathematical problem, alone and without guidance, then the experience of having to parse and unpack a question, extract its significant meaning, devise a strategy, go down blind alleys, get stuck, take risks, and to wait for inspiration, can combine to produce such a shock that the student simply gives up. It can be a wholly negative experience, which if not addressed would be strongly counterproductive.

The challenge presented by problem-solving is as much emotional (or affective) as intellectual; it helps the teacher to know if students have been exposed to it before and whether they have learnt to negotiate it successfully. If they have, they will deal much better with problem-solving in a university context and can play a role in a group to reassure and suggest strategies for other students to follow.

Pre-university students take a broad range of qualifications and have widely differing mathematical backgrounds, but the majority of applicants to English and Welsh HEIs have standard A-Levels and will have taken one or more of the following examinations:

- A-Level Mathematics
- A-Level Further Mathematics
- Advanced Extension Award (AEA) Mathematics paper
- Sixth Term Examination Papers (STEP) I, II, III administered by the Cambridge Assessment Examination Board

In 2011 almost 83,000 took a full Mathematics A-Level, over 12,000 took A-Level Further Mathematics, and over 1,200 took at least one STEP paper. These figures have been steadily increasing since the Further Mathematics Support Programme went national in 2005. The level of challenge in the exam questions increases from A-Level up to STEP, with STEP being significantly more demanding than the rest. (STEP I and II are based on the core specification for A-Level Mathematics; STEP III is designed for students who have also done A-Level Further Mathematics.)

A good way to get a feel for a student's pre-university experience is to look at recent past papers, which can be downloaded from the websites of the exam boards (A-Levels from AQA, Edexcel and OCR, Advanced Extension Award papers from Edexcel, and STEP papers from the Cambridge Assessment Board website.)

To give the reader a feel for the difference, we will now give typical examples of an A-Level question and a STEP question.

An A-Level Question

Example 2.3.1 is Question 6 on the June 2011 AQA Pure Core 4 Mathematics A-Level paper. The exam lasted 90 minutes and there was a total of 75 marks available on the paper. The following question therefore represents about 13% of the maximum mark and should take about 12 minutes.

▼ Example 2.3.1

A curve is defined by the equation $2y + e^{2x}y^2 = x^2 + C$*, where* C *is a constant. The point* $P = \left(1, \frac{1}{e}\right)$ *lies on the curve.*

1. *Find the exact value of* C*.* *(1 mark)*

2. *Find an expression for* $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ *in terms of* x *and* y. *(7 marks)*
3. *Verify that* $P = \left(1, \dfrac{1}{e}\right)$ *is a stationary point on the curve.* *(2 marks)*

A STEP Question

Example 2.3.2 is Question 5 on the STEP I paper also sat in June 2011. The exam lasted 3 hours, and the paper contained a choice of 13 questions on pure mathematics, mechanics, probability and statistics. At most six questions could be attempted, complete answers to four questions would typically earn a grade 1, and little credit was given for fragmentary answers. The following question therefore represents about 25% of a very good mark and a candidate should expect to devote up to 40 minutes to complete it.

▼ Example 2.3.2

Given that $0 < k < 1$, *show with the help of a sketch that the equation*

$$\sin x = kx \qquad (\dagger)$$

has a unique solution in the range $0 < x < \pi$.

Let

$$I = \int_0^{\pi} |\sin x - kx| \mathrm{d}x.$$

Show that

$$I = \frac{\pi^2 \sin\alpha}{2\alpha} - 2\cos\alpha - \alpha\sin\alpha,$$

where α *is the unique solution of* $\dagger$.

Show that I, *regarded as a function of* α, *has a unique stationary value and that this stationary value is a minimum. Deduce that the smallest value of* I *is*

$$-2\cos\frac{\pi}{\sqrt{2}}.$$

Both examinations from which these questions are taken would normally be taken at the end of a student's final year at school (Year 13). A solution to the A-Level question requires one differentiation, two substitutions, and the knowledge that stationary points of a function are the zeros of its derivative.

A solution to the STEP question involves the following:

- A graphical interpretation of the intersection of the sine curve with a straight line, including the fact that the gradient of $\sin x$ is 1 at the origin. (Also implicit use of the intermediate value theorem, though a formal statement is well beyond the boundaries of A-Level mathematics itself.)
- The observation that $\int_0^{\pi} |f(x)| \mathrm{d}x = \int_0^{\alpha} f(x) \mathrm{d}x - \int_{\alpha}^{\pi} f(x) \mathrm{d}x$ because $f(x)$ is negative in the interval $(\alpha, \pi]$.
- Accurate computation of the definite integrals, the correct substitution for k, and algebraic simplification.
- The calculation of $\frac{\mathrm{d}I}{\mathrm{d}\alpha}$ and its non-trivial factorisation into $\left(\frac{\pi^2}{2\alpha^2} - 1\right)(\alpha \cos \alpha - \sin \alpha)$.
- The proof that the second factor is non-zero on the interval $(0, \pi)$ and that $\alpha = \frac{\pi}{\sqrt{2}}$ is therefore the only zero of $\frac{\mathrm{d}I}{\mathrm{d}\alpha}$ in this interval.
- The proof that this stationary point is a minimum and the final calculation of the minimum value of I.

The point we wish to make here is not the *quantity* of mathematical knowledge needed; the STEP question contains topics and techniques from A-level mathematics. Nothing needed to answer the STEP question is particularly tricky or obscure, however the structure of Example 2.3.2 is quite different to that of the A-Level question. The number of linked steps and the need to make subsequent choices based on an understanding of the meaning of the current work clearly separate the two.

We acknowledge that a student may learn to answer Example 2.3.2 in a similar manner to Example 2.3.1, and that the STEP question is worth roughly twice as many marks as the A-Level question. However, it is clear that students who have only been taught to solve questions such as Example 2.3.1 may find themselves in difficulty if confronted with a question such as Example 2.3.2. Not only does the STEP question above call for a range of different ideas and technical skills, it also requires sheer stamina to carry it through to the end without making slips in calculation and manipulation; under the pressures of a timed exam, such slips are hard to spot and recover from.

Returning to Ausubel's dictum, it is important to ascertain what students know, and how they have been previously taught, to provide to them the best transition into university mathematics. This is of particular importance when the experiences of students in a cohort can range so widely.

2.4 Teaching Problem-solving

Problems have been posed to students since antiquity (Boyer and Merzbach, 1991; Robson, 2008), however the modern problem-solving movement within mathematics education can be traced directly to the work of George Pólya. Pólya's book *How to Solve It* (1945) was perhaps the first to discuss problem-solving as a discipline in its own right and the first to attempt to offer a process for solving mathematics problems. Pólya suggests the following process for finding solutions:

1. Understand the problem.
2. Make a plan.
3. Carry out the plan.
4. Looking back.

Thus Pólya presents problem-solving as an iterative process, terminating only once the reviewed plan is confirmed to work. Pólya's two volume *Mathematics and Plausible Reasoning* (1954a; 1954b) expanded greatly on the more populist *How to Solve It*, including a large number of example problems from a wide range of mathematical topics, and introducing more specific approaches to problem-solving such as *generalising and specialising* (see Section 3.5).

After Pólya the development of strategies for solving problems continued, however all faced a similar issue – the more specific a strategy, the less widely applicable it became, and the closer it moved to case-analysis problem-solving. Begle (1979, p. 145) surveyed the empirical literature on the various processes that they offered and came to the conclusion that:

> “No clear-cut directions for mathematics education are provided by the findings of these studies. In fact, there are enough indications that problem-solving strategies are both problem- and student-specific often enough to suggest that hopes of finding one (or a few) strategies which should be taught to all (or most) students are far too simplistic.

Since Begle's review in 1979 a great deal has been published, and authors such as Schoenfeld (1992) and Mason et al. (2010) have developed Pólya's ideas further. Chapter 3 contains John Mason's discussion on the relevance of Pólya to today's problem-solving teaching. We remain, however, unable to teach problem-solving in the systematic way we would teach a subject like calculus. Most recently, Lesh and Zawojewski (2007, p. 768) summed up the situation as follows:

> “In mathematics education, Pólya-style problem-solving strategies—such as *draw a picture*, *work backwards*, *look for a similar problem*, or *identify the givens and goals*—have long histories of being advocated as important abilities for students to develop. Although experts often use these terms when giving after-the-fact explanations of their own problem-solving behaviors, and researches find these terms useful descriptors of the behavior of problem solvers they observe, research has not linked direct instruction in these strategies to improved problem-solving performance.

This is hardly surprising; indeed, were someone to develop a procedure or set of guidelines that enabled students to solve a particular family of problems, much as they are taught techniques for integration, then those problems would be transformed to *exercises*. This illustrates, once again, the importance of context for our notion of ‘problem’.

Despite the difficulty of teaching problem-solving, its importance in developing mathematical thinking is appreciated by many, and furthermore the process that mathematicians go through when solving problems can be retrospectively described in common terms (Lester and Kehle, 2003, p. 507). One possible solution to the seeming impasse is to refer back to Pólya’s *Problems and Theorems in Analysis* (1925), co-authored with Szegö, in which it is said that “the independent solving of challenging problems will aid the reader far more than the aphorisms which follow, although as a start these can do him no harm”.

We conclude that teaching mathematical problem-solving should centre on problems that stimulate students’ imaginations, that appeal to their curiosity, that accord with their knowledge and skills, and that present a challenge without appearing Sisyphean. As well as exercising careful judgement in the selection of problems, the teacher’s most important role is to offer just the right amount of support and encouragement, ensuring that a solution is within a student’s grasp but not too easily discovered. In the following section we consider how problems might be chosen—or authored—to make this possible, and to give students the maximum educational benefit from their problem-solving experience.

2.5 A Good Problem

First and foremost, a good problem should be *pedagogically* worthwhile; it should advance students’ knowledge, deepen their understanding, and be relevant to their curriculum. It should also be carefully matched to their ability, challenging them, and yet offering a fair chance of success within a reasonable time. What constitutes a ‘reasonable time’ is at the discretion of the teacher; in the problem-solving module at Queen Mary for instance, each student is given a single problem each week (see Section 7.5). In a problem-solving class, there will usually be a considerable spread of ability and confidence, which brings the risk that some students might never be able to solve a problem on their own. Here, the teacher has an important role in ensuring that all students derive something educationally valuable from the experience.

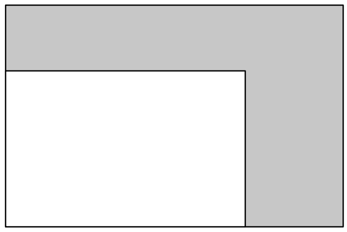
Before discussing examples, we consider some intrinsic features of a mathematical problem that will serve to make it pedagogically worthwhile. We venture that a good problem will embody at least one, and probably more, of the following qualities:

- Manifest aspects of novelty discussed in 2.1.1.
- Bring together several mathematical ideas from different contexts, particularly if this is in an unlikely or surprising way.
- Illustrate the application of one or more general principles that have wider application (e.g. the pigeonhole principle).
- Lead somewhere, being capable of prompting new, related questions, suggesting generalisations or transformations.
- Be capable of reformulation or translation into an equivalent problem with a known solution.
- Form part of a sequence of questions leading to a significant piece of mathematical knowledge.
- Call for some creative insight or flash of inspiration.
- Admit several different solutions, possibly varying in their simplicity or elegance.

Our first example is elementary and looks more like a puzzle than a problem. However, experience has shown that many students struggle with it. It is novel to most of them, it calls for some elementary geometrical insight, and it ‘leads somewhere’:

▼ Example 2.5.1 — *Field*

You own a rectangular piece of land as shown in the plan below. The L-shaped grey region is woodland, the rectangular white part is pasture.



Using a straight-edge only, show on this plan how to build a single straight fence which divides the area of the woodland in half, and justify you answer.

The solution depends on the fact that for a given geometrical shape (a rectangle in this case) there exists a point with the following property: any line through the point divides the area of the shape in half. This leads naturally to the question: which shapes have this property?

Example 2.5.2, taken from group theory, illustrates starkly that novelty is a relative concept. Given the questions studied by Clement et al. (1981) and Wason (1968), it is important to keep in mind students' difficulties with even the slightly unfamiliar and to remember that each year the new cohort are likely to face the same difficulties afresh. A student who has only just met the notation for a coset and the definition of a normal subgroup may well struggle to know where to begin. We would expect that after some active work with tasks of this kind they will soon appreciate the question as an exercise.

▼ Example 2.5.2 — *Centre of a Group*

The centre $Z(G)$ *of a group* G *is defined as follows:*

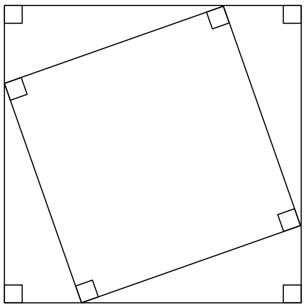
$$Z(G) = \{z \in G \mid zg = gz \;\forall\, g \in G\}.$$

Prove that $Z(G)$ *is a normal subgroup of* G.

Example 2.5.3 challenges students to find a new proof, at least probably new to them, of Pythagoras' Theorem, a result that many of them will have taken for granted since their GCSEs. By being asked to consider familiar material in fresh ways, students will learn there are different, equally valid, routes to the truth, and to make connections between geometric and algebraic approaches to problem-solving.

▼ Example 2.5.3 — *Pythagoras' Theorem*

Making reference to the following diagram, prove Pythagoras' Theorem.



The algebraic solution comes from labelling the two short sides of the four congruent right-angles triangles a and b, say, noting that the large square has area $a^2 + b^2 + 2ab$ and that the combined area of the four triangles is $2ab$. The geometric proof is strikingly visual: rotate the left of the two bottom triangles through 90° about their common point to form a rectangle in the bottom right-hand corner of the large square. Repeat this with the top two triangles to form second rectangle, and finally slide this rectangle across to the top left-hand corner.

▼ Example 2.5.4 — *Triangle*

Five points are placed inside an equilateral triangle with sides of length 2. Show that two of them are less than 1 apart.

The problem in Example 2.5.4 is straightforwardly solved when a well-known technique–namely the pigeonhole principle—is applied. Here, a student is either introduced to new method of proof (if they did not know it before), or extends the range of questions to which they can apply pre-existing knowledge.

▼ Example 2.5.5 — *Homomorphism*

Let $G = GL_n(\mathbb{R})$ denote the set of $n \times n$ invertible matrices over the real numbers. Prove that G, with the operation of ordinary matrix multiplication, is a group. If H denotes the multiplicative group $(\mathbb{R}^*, \cdot)$ of non-zero real numbers, find a non-trivial homomorphism from G to H, justifying your answer.

As our last illustration of a good problem, Example 2.5.5, requires students to bring together knowledge from two different areas of mathematics, group theory and linear algebra, to find a solution. A student totally new to the subject matter may find it hard to make the desired connection, but when they succeed they will be able to view the determinant as a structure-preserving map, rather than just a procedure for calculating a number associated with a square matrix.

The final aspect of a good problem is that students should *want* to solve it. A good problem should somehow intrigue, captivate, and engage the attention of a lively mind; how a problem achieves this is difficult to capture and depends on the curiosity of the student. Our small selection of examples of 'good problems' cannot demonstrate the variety of problems available to test and teach students across an entire mathematics degree. Section 5.4 highlights sources of other suitable problems, and our project website[2] features a collection of its own, to which readers' contributions are welcome.

We do not suppose that a good problem need have all of the features we have discussed in order to be an effective instrument of learning and we emphasise, once again, that a problem's value is a function not only of its content and appeal but also, crucially, of the student for whom it is intended. In the next section we consider how problems can be used as the predominant medium for teaching and learning an area of mathematics.

2.6 Problem-based Learning and the Moore Method

Problem-based Learning (PBL) refers to a range of pedagogies which aim to teach a topic through a progression of problems. PBL is used in a range of subjects and the structure of a problem progression lends itself well to mathematics. Problem-based learning is sometimes conflated with and sometimes seen as distinct from *enquiry-based learning* (EBL, also known as inquiry learning, IL, in the U.S.). In EBL students are presented with tasks by the teacher and by working through these tasks acquire the knowledge that the teacher is trying to convey. In exploring the difference or otherwise between EBL and PBL, Hmelo-Silver et al. (2007, p. 100) had "not uncovered any dimensions that consistently distinguish between PBL and IL".

An extreme form of PBL in mathematics is the *Moore Method*. Named after the influential Texan topologist Robert Lee Moore (1882–1974) (Parker, 2005) who developed it for his university mathematics courses, the Moore Method requires that students answer questions that accumulate into a coherent theory. The essence of the Method is to engage students via the following process:

1. Mathematical problems are posed by the lecturer to the whole class.

[2] www.mathcentre.ac.uk/problemsolving

2. Students solve the problems independently of each other.

3. Students present their solutions on the board to the rest of the class.

4. Students discuss solutions to decide whether they are correct and complete.

5. Students submit written solutions for assessment.

The only written materials that students receive are the problems themselves; no model answers are distributed by the teacher at any time. Teachers are not passive observers however, it is their job to encourage those who are stuck, to ask searching questions and give minor suggestions where they deem them suitable.

Moore has a reputation for running his classes in an authoritarian way. For example, he required that students worked alone; those who sought help from their peers or the published literature were expelled (Parker, 2005, p. 267). One misconception regarding Moore's Method is that he simply stated axioms and theorems and expected students to expound a complete theory. W. Mahavier said of Moore:

> “Moore helped his students a lot but did it in such a way that they did not feel that the help detracted from the satisfaction they received from having solved a problem. He was a master at saying the right thing to the right student at the right time. – Parker, 2005

Moore was particularly successful in attracting and encouraging graduate students, many of whom adopted his teaching approach. As a result, this method is still used widely.

Selecting individual problems for a course taught by such a method is necessary but insufficient for success. Here, it is the *progression* of problems that is key. With a set of problems leading one into another, students build up an area of mathematics for themselves and, as mathematicians do, draw upon previous results to help them achieve this goal. The aim of this approach is not to teach a topic in a polished and professional format, and progress is generally far slower than one would expect from a standard lecture-based module; however these courses pay dividends to students in their other modules, besides teaching them to be better problem solvers (Badger, 2012).

The Moore Method has been extensively modified by practitioners who came after Moore, and arguably only Moore himself used the ‘Moore Method’. The key defining feature here is the progression of problems that build to form a coherent whole. Certain areas of pure mathematics, such as geometry and graph theory, lend themselves particularly well to this approach, however most topics are amenable to it. Variations in students' prior knowledge can disrupt the process (Moore dismissed any student with previous experience in the mathematical topic that he was teaching), and so areas completely unfamiliar to students may prove more fruitful than others.

When teaching students by PBL the temptation may be to lower the initial cognitive burden on students by demonstrating a general approach, before encouraging students to try for themselves, giving fewer and fewer hints as they progress. This *cognitive apprenticeship*, as described by Brown et al. (1989, pp. 455–456), proceeds as follows:

> “ Apprentices learn these methods through a combination of... what we, from the teacher's point of view, call modelling, coaching, and fading. In this sequence of activities, the apprentice repeatedly observes the master executing (or modelling) the target process, which usually involves a number of different but interrelated subskills. The apprentice then attempts to execute the process with guidance and help from the master (coaching). A key aspect of coaching is the provision of scaffolding, which is the support, in the form of reminders and help, that the apprentice requires to approximate the execution of the entire composite of skills. Once the learner has a grasp of the target skill, the master reduces his participation (fades), providing only limited hints, refinements, and feedback to the learner, who practices by successively approximating smooth execution of the whole skill.

Scaffolding is a term introduced by Wood et al. (1976, p. 90) to describe a "process that enables a child or novice to solve a problem, carry out a task or achieve a goal which would be beyond his unassisted efforts. This scaffolding consists essentially of the adult 'controlling' those elements of the task that are initially beyond the learner's capacity, thus permitting him to concentrate on and complete only those elements that are within his range of competence". The processes described by Wood et al. (1976) is termed *guided exploration*, and while appealing in its clarity, presents a number of problems. The first and most obvious problem is that students emulate the teacher's behaviour, without reproducing their approach and reasoning; exactly the issue that we wish to avoid by teaching with problem-solving. Secondly, it is difficult for teachers to maintain control of the complete process, "because a teacher acts in the moment; it is only later that the learners' behaviour makes it possible to describe the whole process as scaffolding and fading. Put another way, 'teaching takes place in time; learning takes place over time' " (Mason et al., 2007, p. 55).

The Moore Method and generalised problem-based learning pedagogies are the basis for the recommendations we make in Chapter 5, and is in use in the mathematics departments at Birmingham (Section 7.1) and Leicester (Section 7.3).

Chapter 3

Having Good Ideas Come-To-Mind: Contemporary Pólya Based Advice for Students of Mathematics

John Mason

Everybody wants students to become better at problem-solving. But what does this actually mean in practice, and what have we learned since George Pólya published his ground-breaking little book *How To Solve It* in 1945? See Appendix B for a potted summary of his advice. Pólya reintroduced the notion of heuristic as the study of the means and methods of problem-solving (Pólya, 1945, p. vi), taken from his free-rendering of a work of Pappus (c300CE) called the *Analyomenos* and augmented by the writings of Bernard Bolzano (1781–1848). Pappus distinguished between *synthesis* (starting from the known and deducing consequences) and *analysis* (starting from the wanted and seeking consequences and connections until reaching a suitable point for synthesis).

Problem-solving has come around yet again as a slogan for reforming teaching of mathematics. The long and the short of the matter is that students in the main continue to assent to what they are told in lectures and tutorials, rather than to assert and subsequently modify conjectures (Mason, 2009). When asked what more support they would like, students usually request more examples, either worked out so as to serve as templates, or for additional practice. They seem content simply to get through the next test, as if they thought that attempting the tasks they are set and scoring passably on tests means that they are learning.

As Pólya says in a preface:

> “This book cannot offer you a magic key that opens all doors and solves all problems, but it offers you good examples for imitation and many opportunities for practice. . . . if you wish to derive most profit from your effort, look out for such

> features of the problem at hand as may be useful in handling the problems to come. . . – Pólya, 1962, p. 1.v[1]

Learning to solve problems not only requires engaging in problem-solving but in addition, intentional acts of learning from that experience.

As long as lecturers and curriculum designers collude by superficial coverage of a multitude of complex ideas without acknowledging the experience of a majority of students, numbers of students pursuing mathematics are likely to continue at a suboptimal level, and suitable students will look elsewhere for a fulfilling career. At the core of the ‘problem’ for students is having come-to-mind an appropriate action to undertake in order to resolve some problematic situation, and it is problematic for them, because they are not aware of how to study mathematics, nor how to think mathematically. In short, they need Pólya’s advice. At the core of the ‘problem’ for lecturers is having come-to-mind an appropriate pedagogic strategy[2] or didactic tactic that will stimulate student learning effectively. Of particular significance is the often over looked advice to ‘look back’ on what you have done and try to learn from the experience. Pólya presents his advice experientially, through inviting readers to try things for themselves, then watch an expert and then try more things for themselves. The case is made here that Pólya’s age-old advice and style of presentation remains the best there is, being highly pertinent and readily augmented by research and experience garnered subsequently.

3.1 The Many Meanings of Problem Solving

Political phrases gain potency for a time by being multiply construable and Problem Solving is an excellent case in point. It has different meanings in different settings as indicated in Chapter 2. As a commonplace ‘problem-solving’ resonates with widely varying audiences; as a practice it is highly attenuated. Pólya addresses the issue of trying to specify what he means by ‘problem-solving’, arriving eventually at:

> “ To search consciously for some action appropriate to attain a clearly conceived but not immediately attainable aim. – Pólya, 1962, p. 1.117

This conforms with an insight expressed in the 1960s by Brookes (1976) in a short article called *When is a problem?*, pointing out that something is a problem only when there is a person for whom it is a problem, and that what is a problem for one person may not be for

[2]A *pedagogic strategy* can be used in many different settings and mathematical topics; a *didactic tactic* is specific to the topic, concept or technique.

another. Pólya (1962, p. 2.63) also addresses the importance of desire in generating a state of 'problématique'.

Pólya proposes two kinds of mathematical problems and discusses differences in approaches required: problems *to prove* and problems *to find* something (and then prove) (Pólya, 1962, pp. 1.119–121).

3.1.1 Word Problems

Most notable among the exercises in traditional textbooks are the so-called 'word problems'. These are routine problems set in some usually fictitious if not fanciful context. Isaac Newton was one of the notable figures who redirected mathematicians' attention from the solution of 'puzzles' to the solution of the equations that you get from tackling puzzles and word problems (see Whiteside, 1968). He thought the translation or 'modelling' aspect was essentially trivial, although his contemporaries were not so sure, and they have been borne out by the notable failure of students in subsequent generations to master such 'problems'.

Much maligned as a genre (Gerofsky, 1996; Verschaffel et al., 2000), word problems seen not as effective modelling of the material world, but as opportunities to discern pertinent details, to recognise relevant relationships and to express these as properties in symbols and diagrams so as to use algebraic and other techniques, can serve a useful purpose (Mason, 2001). Even so, this is not what is meant by *problem-solving* at tertiary level, although it would be greatly advantageous if students on entry were better at it.

3.1.2 Problems as Exercises

In some countries 'problem-solving' usually refers to students working on the exercises at the ends of chapters. These can vary according both to the degree to which they extend and develop the chapter contents, and the degree to which they are structured so as to reveal underlying relationships. While it is tempting to classify tasks according to their degree of internal structure, and the degree to which they provide practice rather than directions for exploration, structure in a set of exercises involves lies partly, even largely, in the eyes of the beholder. What matters is the way the exercises are used. Here are some examples:

Carefully Structured

Example 3.1.1 is taken from Krause (1986, p. 6). The pedagogic subtlety can only be experienced by actually doing the task.

▼ Example 3.1.1 — *Krause*

For this exercise $A = (-2, -1)$. Mark A on a coordinate grid. For each point P below calculate $d_T(P, A)$ [the taxicab distance], and mark P on the grid [in the original, they are in a single column so there is no temptation to work across rows instead of in order down the columns]:

(a)	$P = (1, -1)$	*(e)*	$P = (\frac{1}{2}, -1\frac{1}{2})$
(b)	$P = (-2, -4)$	*(f)*	$P = (-1\frac{1}{2}, -3\frac{1}{2})$
(c)	$P = (-1, -3)$	*(g)*	$P = (0, 0)$
(d)	$P = (0, -2)$	*(h)*	$P = (-2, 2)$

Here there are lurking surprises that provoke re-thinking expectations, leading to a deeper sense of how distance 'works' in an unfamiliar metric. It may look like routine practice, but it is actually highly structured and develops the concept of 'circle'.

From a sixth form pure mathematics textbook (Backhouse et al., 1971, p. 59) which has been revised and reprinted many times right up until 2011:

▼ Example 3.1.2 — *Backhouse*

1. *Express* $\frac{2}{n(n+2)}$ *in partial fractions, and deduce that*

$$\frac{1}{1.3} + \frac{1}{2.4} + \frac{1}{3.5} + \ldots + \frac{1}{n(n+2)} = \frac{3}{4} - \frac{2n+3}{2(n+1)(n+2)}.$$

2. *Express* $\frac{n+3}{(n-1)n(n+1)}$ *in partial fractions, and deduce that*

$$\frac{5}{1.2.3} + \frac{6}{3.4.5} + \frac{7}{3.4.5} + \ldots + \frac{n+3}{(n-1)n(n+1)} = 1\frac{1}{2} - \frac{n+2}{n(n+1)}$$

5. *Find the sum of the first n terms of the following series:*

(i)	$\frac{1}{1.4} + \frac{1}{2.5} + \frac{1}{3.6} + \ldots,$	*(iv)*	$\frac{1}{2.6} + \frac{1}{4.8} + \frac{1}{6.10} + \ldots,$
(ii)	$\frac{1}{2.4} + \frac{1}{4.6} + \frac{1}{6.8} + \ldots,$	*(v)*	$\frac{1}{1.3.5} + \frac{1}{2.4.6} + \frac{1}{3.5.7} + \ldots,$
(iii)	$\frac{1}{3.6} + \frac{1}{6.9} + \frac{1}{9.12} + \ldots,$	*(vi)*	$\frac{1}{3.4.5} + \frac{2}{4.5.6} + \frac{3}{5.6.7} + \ldots.$

Questions 3 and 4 make the transition to sums of series.

Here the technique of partial fractions is used to rewrite series as telescoping sums. Either the student is expected to generalise for themselves, and to extract if not abstract a technique, or the teacher is expected to draw student attention to this. Although any mathematician would expect students to make this generalisation and reach this abstraction for themselves, it would be perfectly possible for a student to do all of question 5 successfully without actually becoming aware of an underlying structural technique.

In his *Higher Algebra for Schools* (1941, pp. 6–7) Ferrar shows two worked examples using telescoping terms to find the sum of a series, and then adds the comment: "Pay attention to the idea not the particulars" (p. 13). However, in a later edition (1956) this is removed and the presentation revised. The comment is changed to:

> " the reader should concentrate on the procedure 'express u_r as the difference of two v_r's; add $u_1 + \ldots + u_n$ and see which terms cancel'; do not attempt to remember the precise way in which they cancel in each type of example. – Ferrar, 1956, p. 9

it seems that perhaps Ferrar came to the conclusion that his indirect advice was insufficient, and that he needed to be more specific. Whenever we are specific, we take away some of the potential for students to take initiative; whenever we leave things unsaid, students may overlook the essence. Prompting reflection is one way to work in the middle ground between the two extremes.

Interestingly, in his *Higher Algebra* sequel (1943, p. 96) Ferrar inserts comments in square brackets, warning the reader that "they do not form a necessary part of the solution when the reader is working examples for himself", which suggests that his experience corresponds closely to Open University experience, that students try to 'learn' and then 'display' whatever is in their text, rather than distinguishing advice from necessary behaviour. Elsewhere in his two books is there a similar comment about where attention is best directed. Presumably students, or students' teachers are expected to recognise and point out these matters.

Learning from examples

Ference Marton (see Marton et al. (1997); Marton and Pang (2006); Marton and Säljö (1976a,b); Marton and Trigwell (2000)) suggests that what is available to be learned from a sequence of examples, and by extension, from a sequence of exercises, is what is varied. He uses the terms *dimensions of variation* to refer to the aspects which are varied, and observes that to be detected, variation needs to take place within a confined space and time. Thus it is of little use to sprinkle instances amongst differently structured ones (as in a set of miscellaneous problems) if the intention is that students appreciate the possible variation and the *range of permissible change* in each of these 'dimensions' (Watson and Mason, 2005, 2006). For example, recognising numbers as potential parameters is an act of generalisation. But it is perfectly possible to treat a worked example as a template without really appreciating what is being varied and over what range that variation is permitted, and hence without actually experiencing the abstraction (Pólya, 1945, pp. 209–214).

Scataglini-Belghitar and Mason (2011) report on an analysis task involving the use of the theorem that a continuous function on a closed and bounded interval attains its extreme values. Even though students had seen an application of the theorem, many did not think to use it on a test when they had to construct an appropriate interval for themselves, nor did it come to mind for them in a subsequent tutorial. It can be difficult for an expert to recognise that students may not be attending to the same things, or even if attending to the same things, not attending in the same way (Mason, 2003). Thus the tutor experiences an instance of a general property, while students most often experience a particular. Familiarity tends to increase speed of talk and decrease verbal and physical pointing, reducing the opportunity for listeners to construe what is being said.

More challenging but less structured

▼ Example 3.1.3

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12. If $z_1, z_2, \ldots z_n$ are complex, prove that $|z_1 + z_2 + \ldots + z_n| \leq |z_1| + |z_2| + \ldots + |z_n|$.

13. If x, y are complex, prove that $||x| - |y|| \leq |x - y|$.

14. If z is a complex number such that $|z| = 1$, that is, such that $z\bar{z} = 1$, compute $|1 + z|^2 + |1 - z|^2$. — Rudin, 1976, p. 23

These form a somewhat isolated trio of tasks in an end of section set of exercises concerning the modulus sign but with no evident structural relationship apart from 12 and 13 being important inequalities that are used frequently, while 14 demonstrates the power of using complex conjugates when calculating with lengths of complex numbers. Yet there is nothing to alert students to these facts as useful properties.

The three exercises in Example 3.1.4 are taken from Herstein (2006, p. 46), a textbook that has been in use for some 50 years.

▼ Example 3.1.4 — *Herstein*

1. If H & K are subgroups of G, show that $H \cap K$ is a subgroup of G. (Can you see that the same proof shows that the intersection of any number of subgroups of G, finite or infinite, is again a subgroup of G?)
2. Let G be a group such that the intersection of all its subgroups which are different from (e) is a subgroup different from (e). Prove that every element of G has finite order.
3. If G has no non trivial subgroups, show that G must be of finite prime order.

These problems form a structured sequence, but are students likely to recognise that structure as indicating a strategy for proving a result for themselves, without having their attention drawn specifically and explicitly to how the problems are related?

Subsequent 'exercises' are used to introduce cosets, centre and normalizer, asking the student to go through the (for an expert, trivial) reasoning that would otherwise use up several pages of text. The mathematically inclined student may shift from recognising relationships in a particular task, to perceiving these as instances of properties which are the actual focus of the task, while students not yet enculturated into mathematical thinking might overlook their import and so not actually learn what is intended.

Unstructured

The following exercise set is taken from Lang (1987, p. 117).

▼ Example 3.1.5 — *Unstructured questions*

In each of the following cases, write $f = qg + r$ with $deg(r) < deg(g)$:

$$\begin{array}{lll} (a) & f(t) = t^2 - 2t + 1 & g(t) = t - 1 \\ (b) & f(t) = t^3 + t - 1 & g(t) = t^2 + 1 \\ (c) & f(t) = t^3 + t & g(t) = t \\ (d) & f(t) = t^3 - 1 & g(t) = t - 1 \end{array}$$

In fact there is no need to use the Euclidean algorithm as they can all be done by inspection if you know any simply factoring. Thus the exercises do not offer practice in using the algorithm. Furthermore the remainders of all but (b) are 0. Perhaps this is designed to raise the question of the remainder being 0 if and only if g divides f, but because of the limited variation, there is no indication of the remainder theorem, nor any indication of how the same algorithm can be used to find the greatest common divisor of two polynomials.

3.1.3 Problems as Consolidation

In many universities, students are assigned weekly 'problem sheets' which are then corrected and-or commented upon in tutorials, whether of the one-to-a-few or one-to-many variety. Often these tasks require students to understand and appreciate definitions and theorems mentioned in lectures, applying them in what are for students, novel contexts. Nevertheless, they most often build on or activate book-work from the course. Past experience in school of using worked examples as templates is rarely effective for the problems at university level which often require or are intended to provoke appreciation of a technique, theorem or concept.

The following is taken from *Analysis II*, an Oxford course:

▼ Example 3.1.6

6. [Optional] Suppose that the conditions for the Mean Value Theorem hold for the function $f : [a, a+h] \to \mathbb{R}$, so that for some $\theta \in (0,1)$ we have $f(a+h) - f(a) = hf'(a+\theta h)$. Fix f and a, and for each non-zero h write $\theta(h)$ for a corresponding value of θ.

Prove that if $f''(a)$ exists and is non-zero then $\lim_{h\to 0} \theta(h) = \frac{1}{2}$. Need this be true if $f''(a) = 0$?

Here students are presumably expected to interpret the last part as an invitation to look for or construct a counter example. How many are likely to recognize the import, namely that for a differentiable function, the Rolle-point associated with an interval tends to the midpoint as the interval width approaches zero?

3.1.4 Problems as Construction Tasks

Students, particularly but certainly not exclusively those in applied disciplines such as science, engineering, computer science, and social sciences, are minded to pick up techniques like unfamiliar tools in a tool box, and try to make them fit some new situation. Often they use a theorem or associated technique without checking whether the necessary conditions are valid in their situation. One of the ways of addressing this is to invite students to explore the effects of weakening conditions, through the construction of counter-examples. Constructing examples, and counter-examples to slight modifications, is a primary way that mathematicians have for appreciating the force and scope of a theorem and its associated definitions and lemmas.

> “A good stock of examples, as large as possible, is indispensible for a thorough understanding of any concept, and when I want to learn something new, *I make it my first job to build one.* – Halmos, 1983, p. 63

A useful feature of a ‘stock of examples’, otherwise known as a student’s *accessible example space* (Watson and Mason, 2005), is a collection of construction tools which accompany the examples so that they can be modified and tinkered with, rather than being isolated. The richness of the example-space to which a student has access, in terms of links and connections, multiplicity of examples, and associated construction techniques for tinkering with examples is a measure of the depth of understanding and appreciation. The pedagogic issue is whether providing counter-examples in texts or notes is of as much benefit for students as learning to look for and construct their own examples meeting specified constraints, so that they are enculturated into the practice of looking for counter-examples in particular, while accumulating a richly stocked example space.

Examples

- Construct functions to show that
  - a continuous function on a bounded interval need not assume its extremal values;
  - a continuous function on a closed interval need not be bounded;
  - a function on a closed bounded interval need not assume its extreme values.
- Construct a sequence of positive terms converging to 0 with the property that for any n there is a subsequence of length n which is strictly increasing.
- Construct a function for which to calculate the limit at a particular point requires the use of L’Hôpital’s theorem n times.
- Construct a linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ with the property that there is a vector v whose image $T(v)$ is at right-angles to v, but that there are other vectors for which this is not the case.

The first invites students to look at the conditions required in the relevant theorem, and to see for themselves why all the conditions are necessary. The pedagogic issue is whether students realise that they could/should do this for every theorem they encounter. Taking such a stance enriches student appreciation of the concepts and the theorem.

The second challenges the naïve intuition that sequences eventually 'go straight to their limit', with the potential to enrich their personal example space and their concept image for convergent sequences.

The third opens up the possibility of multiple uses of the same technique, enriching students' awareness of the range of application of the technique.

The fourth challenges students' appreciation of eigenvectors by embedding their use in a context that does not have the usual eigen-cues.

3.1.5 Problems as Explorations

Commentators and mathematicians alike often complain about a lack of creativity and initiative among students. However, if students do not see relative experts engaging in mathematical thinking, if they are not immersed in the valued aspects of mathematical thinking as a matter of course, how are they to pick up the practices and engage fully themselves? For example, how often have students watched a lecturer pose a problem spontaneously or vary a definition in order to enrich the associated example space? How often, when this does happen, do students realise what is happening rather than being immersed in the particularities of the situation? If students are expected to display creativity, such as posing problems arising from the topics they are being taught, then they need to be in the presence of experts doing exactly that.

Are students expected to build up a rich example space like filling a pantry with pre-packaged groceries (Watson and Mason, 2005)? Without being exposed to example construction, how are students to discover its central role in mathematical thinking for themselves?

As David Tall and Shlomo Vinner (1981) observed, the rich associations that experts have with technical terms, their *concept images*, go far beyond formal definitions. Furthermore, students are as likely to begin by stressing irrelevant as relevant features as they refine and enrich their concept image to discover what is and is not relevant. How then are students who are exposed to a few 'examples' and a definition supposed to appreciate the scope, significance, and import of a definition? Tasks can be set which prompt students to enrich their space of associated examples.

Examples

- For any subset S of the reals, let $A(S)$ denote the set of accumulation points of S and $B(S)$ denote the boundary points of S (points which are limit points for sequences both entirely in, and entirely outside of S). Characterise $\{A(S) \mid S \subseteq \mathbb{R}\}$ and $\{B(S) \mid S \subseteq \mathbb{R}\}$.

- Let G be a group and $P(G)$ the set of subsets of G. For each subset S of $P(G)$ induce a binary operation on S by $A \circ B = \{ab \mid a \in A$ and $b \in B$ as elements in $G\}$. For which S is $(S, \circ)$ a group?
- Let $\{e_1, e_2\}$ denote the standard basis for $\mathbb{R}^2$ and let $\{f_1, f_2\}$ denote another basis. Let T be a non-singular linear transformation of $\mathbb{R}^2$ to itself. What is the boundary of the region $\mathbb{R}^2$ for which $f_2 \in \mathbb{R}^2$ means that T has real eigenvalues?
- Given a differentiable function f on a closed interval, where on the interval would you look for a Rolle point, where the slope of f is the slope of the chord over the whole interval?
- Which matrices can represent the non-singular linear transformation T from a real-vector space V to itself?

Tasks like these can be used as motivation ("by the end of this section you will be able to address problems like…"). As well as promoting exploration, tasks like the last one can be used as a form of revision: if you really understand linear transformations, no specialising or exploring is necessary.

3.2 The Social Component of Mathematical Thinking

In common with every other discipline, people learn to think mathematically by being in the presence of others who are thinking mathematically, whether through reading written texts or through attending live lectures and participating in support groups. Their sense of what sorts of questions mathematics addresses are informed by the types of questions they are asked to work on. But what are students to make of their experiences of school classrooms, of reading textbooks and attending university lectures and tutorials? The questions they encounter there are largely of the routine, technique-rehearsal variety, or else highly refined exposition. The students' job is to do enough exercises so that they think they can 'do a similar one' on their own in the future when required. A retired chemist complained to me recently that students arrive at (his) university without having practiced algebraic manipulation sufficiently to have internalised and automated it. But is effective practice merely repetition and rehearsal? For example, are students encouraged to try to articulate what 'similar' means, in their own words, when recognising two tasks as similar? Are they encouraged to articulate for themselves the steps in a technique so as to develop their own 'inner incantations'?

Certainly techniques (sometimes referred to as skills) are important, but they are not the essence of mathematical thinking. Even where there is some initial 'motivational' material setting some contexts in which a topic or concept can be or has been applied, and even where there are some consolidation exercises indicating scope of application, student attention is naturally focused on completing the tasks assigned. The natural, if naïve epistemology is 'if I do the tasks I am given, as best as I can, then the intended learning will take place'. The flip side of this *didactic contract* (Brousseau, 1997) is that the teacher assigns tasks which, it is believed, will contribute to the students' internalisation of concepts and techniques. But if there is more to mathematics than definitions, theorems and rehearsing techniques, as most philosophically and educationally minded commentators seem to believe, then it behoves us to develop the desire to think mathematically, the experience of thinking mathematically, and the use of appropriate human powers so as to get pleasure from thinking mathematically.

Pólya is very clear about the role of textbook authors and teachers: "teachers and authors of textbooks should not forget that the intelligent student and the INTELLIGENT READER are not satisfied by verifying the steps of reasoning are correct but also want to know the motive and the purpose. . . . Mathematics is interesting in so far as it occupies our reasoning and inventive powers." (Capitals in the original *How to Solve It* (1945, p. 49)).

3.3 Issue of Time

It is a reasonable conjecture that for as long as people have thought about the teaching and learning of mathematics (some 4000 years at least), the issue of time has been uppermost: the time required for students to explore around a definition or a topic effectively is bound to reduce the amount of content to which they can be exposed. One extreme considers it necessary to provide a rich if superficial exposure to many ideas which students can subsequently work both on and through, over an extended period of time. At the other extreme, perhaps significant learning will only take place when students struggle for themselves with the core concepts and theorems (see Section 2.6 on the Moore Method).

Neither extreme is necessary. The advantages and contributions of both can be obtained. But to achieve this, students need to be enculturated into thinking mathematically, not simply left to their own devices to try to internalise the concepts and theorems and ways of thinking to which they are exposed. Put another way, if lecturers aim *to be mathematical, with and in front of their students*, then there is a good chance that many will pick up some of the practices and develop their own mathematical thinking. If these practices remain part of the *hidden curriculum* (Snyder, 1973) that students are expected to adopt without explicit reference, then only the most mathematically attuned students will survive, and many others may actually be put off further study. It is for all of the reasons explicitly and implicitly alluded to here, that it is helpful to refer to *promoting and sustaining mathematical thinking* rather than to *problem-solving*.

3.4 Promoting and Sustaining Mathematical Thinking

George Pólya was a major force in the recognition of mathematics education as an important aspect of mathematics: if students are not encouraged, intrigued and engaged, then mathematics itself will suffer. His two books, available in a single volume (1962) initiated a major strand of mathematics education based on experience + reflection. In his first book, *How To Solve It* (1945, p. 49), he advanced four stages:

Understanding the Problem; Making a Plan; Carrying Out the Plan; Looking Back.

As advice, this is pretty bland and difficult to use when you get stuck. Jim Wilson, a student of Pólya's, once observed (private communication), that Looking Back is the stage most talked about and yet most notably absent from mathematics classrooms. For example, Pólya uses the metaphor of mushrooms: "Look around when you have got your first mushroom or made your first discovery; they grow in clusters" (1962, pg 1.225). Not only is it worthwhile to check the reasoning (1945, p. 15), but it is even more important to ask whether the result can be derived differently, perhaps more elegantly or more simply, or even, when looked at appropriately, 'seen at a glance'. And perhaps most important of all is to use 'looking back' to look forward by asking yourself whether it might be possible to use the result in some other situation (1945, pp. 15–16)?

It is not that there is a complex practice involved, but even where tasks are set such as the ones presented earlier as 'construction tasks', for many students it will be necessary to provoke them to withdraw from being immersed in the activity, and to focus their attention on what the task demonstrates, and on how it instantiates a valuable if not essential study-practice. Pólya's subsequent books provide a more systematic and comprehensive insight into what the four stages can mean in practice.

Every teacher has encountered students who, on being set a task, immediately start doing the first thing that comes to mind. Pólya (1962, p. 2.34) drew attention to the value of pausing, of parking the first idea and seeing if something else more effective or insightful might also come-to-mind. Pólya, in typical fashion uses adages or proverbs such as "look before you leap", "try before you trust" and "a wise delay makes the road safe" (1945, p. 224). The range of expressions is presumably in the hope that one or other will strike a chord and so be available to come-to-mind when needed in the future. Beneath the gamut of advice, heuristics and processes that he identified, effective and creative thinking boils down to whether, when needed, some pertinent action comes-to-mind, and this theme permeates his books. For most people most of the time, this depends on enacting Pólya's last stage: Looking Back.

Pólya's two-volume work is full of mathematical 'problems' which he chose in order to illustrate and provide experience of various aspects of mathematical thinking and problem-solving. His legacy can be found in Olympiad training (for example, Holton (2009, 2011)), and in the work of noted authors such as Jeremy Kilpatrick, who was Pólya's first mathematics education Ph.D. student, Alan Schoenfeld (1985), and Tony Gardiner (1987) among many others. When the Open University (U.K.) began its mathematics courses, it chose *How to Solve It* as a set text, then developed a strand of mathematical investigations illustrating Pólya's advice in its summer schools. This was articulated in *Thinking Mathematically* (Mason et al., 2010) and *Learning & Doing Mathematics* (Mason, 1988), which were intended to make Pólya's ideas accessible to undergraduates and to pre-service and in-service teachers. The ideas were integrated into many Open University courses and texts.

Pólya drew specific attention to processes (in the language of the 60s and 70s) such as specialising and generalising, conjecturing and convincing, and the use of heuristics such as 'try working backwards' (Pólya, 1945, pp. 225–232), etc.. After working with these for a long time, and as the language of 'processes' became less attractive, it became evident that Pólya had been tapping in to fundamental human powers that underpin human activity. Because it seems necessary to re-articulate insights in the vernacular of the day, and because it helps to integrate them if you re-express them for yourself, the list was augmented to the natural powers of:

- Imagining & Expressing
- Specialising & Generalising (Stressing & Ignoring)
- Conjecturing & Convincing
- Organising & Characterising
- Restricting & Expanding

while the heuristics (someone accumulated 99 of them) were downplayed and replaced by a focus on mathematical themes, such as:

- Doing & Undoing
- Invariance in the Midst of Change
- Freedom & Constraint

A brief amplification of each of these powers and themes is in order, because it was a shock to discover that many Open University tutors were not sufficiently aware of their own use of these to have them come-to-mind explicitly when working with students. Consequently, student attention was not being drawn to them explicitly as often as desirable. Thus while it may be obvious that these processes contribute to mathematical thinking, it is worth elaborating them so that their role comes-to-mind when teaching, and so that ways in which students can be encouraged, even stimulated to use these powers for themselves, can be integrated into any style of mathematics teaching. When working with students, nothing is as effective as engaging students in a task in which they spontaneously use a power for themselves, so that attention can then be drawn to what they have already experienced naturally.

3.5 Powers

The notion of exploiting human powers, because intellectual pleasure comes from the use and development of one's own powers, has roots in the work of many authors, notably Herbert Spencer (1864, p. 72) and Caleb Gattegno (2010, p. 165).

Imagining & Expressing

Imagining what is not physically present is a fundamental power that makes planning possible, for it is how humans direct their emotional energies to achieving goals. Here the word *image* is used in its broadest meaning, incorporating all of the human senses. In mathematics it refers to any 'sense-of' relationships and properties, whether in pictorial, symbolic or more diffuse intuitions. Expressing what is imagined using whatever medium seems appropriate (sketches, diagrams, words, symbols, gestures, voice tones, and displays on electronic screens) is the basis for social interaction, through which we amplify, modify, and attenuate our conjectures. Mathematical modelling makes core use of these powers as the intermediate phase between encountering a problem and preparing a mathematical model of the situation.

Getting students to express (if only to themselves or to each other) what they understand by a concept (aspects, particularly images forming part of their developing concept-images) or the way a technique works, or what is the same and different about various mathematical objects, including exercises, is a valuable study technique which can be learned from being activated in lectures and tutorials.

Stressing & Ignoring

Gattegno (2010) noticed that generalisation comes about through stressing some features and consequently ignoring others. Abstraction involves isolating properties and then stressing the properties while downplaying (if not ignoring) any examples from which they were extracted. Pólya connects abstraction with simplification, drawing on Descartes' advice to strip away all irrelevant detail (Pólya, 1962, p. 1.50).

One of the things that a lecturer or tutor can do is to amplify what needs to be stressed, thereby attenuating attention on what needs to be ignored, in order to promote appropriate generalising and abstracting.

Specialising & Generalising

Specialising (or particularising) means much more than simply trying an example, an instance of something more general (Pólya, 1945, pp. 108–109, 190–196; Pólya, 1962, pp. 1.78, 1.178). The aim of specialising is to recognise structural relationships, and to begin to see these as instances of properties worth perceiving in different situations, that is, to (re)generalise for oneself. It is through 'seeing the general through the particular' that we encounter generalisation (through stressing and ignoring) and it is through 'seeing the particular in the general' that we test intuitions and contact structural relationships that might lead to justifications of conjectures.

Under the chapter heading of 'superposition', Pólya (1962, pp. 1.99–113) displays instances of 'varying the data' so as to consider particular (special) cases which then inform or can even be assembled or superimposed to yield what was originally an apparently difficult generality. For example, Lagrange's interpolation formula and the circle theorem that the angle subtended at the centre is twice the angle subtended at the circumference can both be attacked using this principle.

By promoting the actions of specialising and generalising in concert, lecturers can promote students' use of their own powers to make sense of mathematics, and to make mathematical sense of situations. If concepts are always introduced through examples, or always introduced through a definition followed by examples, students are likely to become dependent on the expert for the specialising and generalising that they could be doing for themselves. Pólya advises "observe, generalize, prove and prove again" (more efficiently) (1962, p. 1.76).

Conjecturing & Convincing

Being aware of the status of an assertion is vital in mathematics. If students are immersed in a conjecturing atmosphere in which claims are pondered, tested and often modified, then mathematical thinking can flourish; if they are immersed in an atmosphere that values only correct answers and insightful remarks, many dimensions of the human psyche are likely to become atrophied in relation to mathematics. Pólya (1962, pp. 1.143–168, 2.156–157) extolls the virtues of 'guessing' but then not believing your guess.

Learning to reason mathematically is a core aspect of studying mathematics. This involves learning to convince yourself, then a friend, then a sceptic (Mason et al., 2010) each more demanding and more critical than the last, on the way to developing a robust proof. Learning to critique other people's reasoning is a valuable contribution to learning to reason for oneself. If students only encounter correct (and distilled, clever, aesthetically pleasing) proofs they are likely to be mislead into assuming that this is how proofs appear. Students are then at a disadvantage because refining a proof is a many-stage process. Simple and clever proofs do not simply flow out of the pen or the chalk at the first attempt.

Learning to reason mathematically also depends on shifting attention from discerning details, to recognising structural relationships in the particular, to perceiving these relationships as properties. Only then does it make sense to try to use only agreed properties to justify conjectures (Mason, 2003).

Organising & Characterising

Human beings are great organisers and classifiers. After all, this is the basis for language, which is inherently general and based on classification. Getting students to use these powers themselves through sorting tasks augmented by constructions of mathematical objects belonging to different classes is an excellent way for them to enrich their personal example spaces and their sense-of concepts and techniques. Most theorems in pure mathematics can be seen as classifications: the objects satisfying the claims are precisely the objects satisfying the conclusions, or at least a subset of them.

Restricting & Extending

Throughout schooling the notion of *number* is extended and extended. By restricting attention or by extending attention, such as restricting domains of functions, or extending domains to higher dimensions, the import of definitions can be enriched, and structural theorems elucidated. The notion of sub-structures such as subspaces, subgroups, subrings etc. arises from restricting attention, and techniques for constructing new objects from old provide examples of extending. Restricting attention to subclasses often increases the range of provable theorems (real-valued functions, continuous functions, differentiable functions etc.) Weakening a constraint or property extends the domain of study so as to include more objects.

3.6 Themes

Themes such as the following pervade mathematics, providing a connecting thread through apparently disparate topics.

Doing & Undoing

Whenever you find that you can perform an action, the question arises of characterising those objects that would give the same result (inverse image). Similarly, asking what other results are possible explores the possible co-domain. Given a technique for 'doing' something gives rise to an undoing: constructing an object with a given result when the 'doing' is carried out. Often an 'undoing' question requires creativity due to multiple possibilities, and poses a difficult challenge (Groetsch, 1999).

Examples

- Given a triangle, finding the circumcentre, or the medians, or the incentre is straightforward. Given the medians (as lengths) finding all triangles with their medians as those lengths is more of a challenge; given the centroid, incentre and circumcentre, finding the triangle is also a challenge.

- Given a Möbius transformation, finding whether it is of finite order is straightforward; constructing Möbius transformations of a given finite order is more of a challenge, and more illuminating.

Whenever you want to do something in a complicated situation which you can do in a simple situation, it makes sense to try to transform the complicated into the simple, perform the action, and then transform back again. Melzak (1983) called this *bypassing* but under the label *conjugation* it is the structure exploited both in and by groups. It is one instance of combining *doing* and *undoing* to resolve a complex problem by making it simpler first.

Invariance in the Midst of Change

Most theorems in pure mathematics can be viewed as stating something that remains invariant while something else changes. Often it is a challenge, even an open problem, to discover what the range of permissible change is while maintaining a given invariant. When students are specialising, they are looking for structural relationships that remain invariant in the midst of the changing parameters of the examples. When they are generalising, they are trying to express these relationships as instances of properties.

When tutors are setting problems, especially when random examples are being generated automatically (Sangwin, 2003; Sangwin and Grove, 2006), they are taking account of structural relationships amongst the parameters that remain invariant as the examples change. Specifying the range-of-permissible change of parameters can be a complex problem in itself.

By being aware of and explicit about invariance in the midst of change, students can link apparently disparate topics together, making it more likely that suitable actions will come-to-mind in the future.

Freedom & Constraint

Most mathematical tasks, from pure calculation to the display of (memorised) bookwork can be seen as constructive activity. Constructing the answer to a calculation can be seen as starting with a mathematical object (perhaps a point, a function, or something more complicated) which has no constraints on it, and then adding constraints. The question is often in the form 'is there an object satisfying these constraints'? By imposing the constraints piecemeal, it is sometimes possible to get a sense of the freedom available with only some of the constraints, an appreciation of the generality, on which further constraints are to be imposed. In that way it is sometimes possible to make progress where a direct approach to satisfying all the constraints at once is too challenging. Seeing things as having freedom which then have constraints imposed enriches students' sense of what they are trying to do in applying a technique to 'solve' a problem.

3.7 Informing Teaching

These reflections suggest that being aware of themes that pervade mathematics could be an important aspect of being a professional mathematician. In particular, making explicit reference to them when they arise could contribute to effective teaching, because it could contribute to student appreciation of how mathematics works, and what constitutes mathematical thinking. Some students notice the *hidden curriculum*, the mathematical practices that make mathematics such a creative, stimulating and fulfilling discipline to follow, but many may not, without having their attention directed to them.

Similarly, being aware of the use of your own powers as a mathematical thinker could inform teaching by looking for ways to provoke students to use their own powers, rather than usurping them through trying to 'do the learning for students'.

The presence, persistence, and practical nature of the processes which Pólya articulated provide a strong foundation for promoting mathematical thinking rather than simply providing support for passing assessment tasks.

Chapter 4

Establishing A New Course – Solving the Problem

Bob Burn

Bob Burn is a mathematician and educator. He has written a number of highly-regarded textbooks for undergraduate mathematics that put problem-solving at the heart of the student's learning. In this chapter, he describes how his book *A Pathway into Number Theory* (Burn, 1982) took shape during his time at at Homerton College, Cambridge. It was first published in 1982.

A space in the curriculum opened for an additional mathematics course for which there was no staffing allocation. We believed the take-up would be modest and thought we could offer supervisions but no lectures. The students would be in the fourth year of a B.Ed. We had wanted the course to be pursued by problem-solving. Our mentors in the university maths department made it clear that there had to be a syllabus and a final examination. So what was the course to be about? After some unconvincing suggestions, a university professor suggested number theory. We thought this was rich in problem-solving potential and agreed that the proposal was promising. But none of us in education had studied a course in number theory, let alone taught one, so how might the course be designed? I volunteered to work on the design of the course provided I could have a contact to talk to in the maths department.

At the time, my volunteering just seemed to be the least worst option, but in retrospect, despite my ignorance of number theory there were aspects of my background which played constructively into the task. Some ten years previously I had completed a Ph.D. in the foundations of geometry. At school and as an undergraduate I had spent my time doing exercises and problems that others had chosen. In the research experience I was tackling questions that I had formulated and was exercised about. The distinction was transformational.

In the research, I had experienced what it was like to make mathematics. The function of proof had changed from something to be memorised for an exam to a struggle for certainty about my own conjectures. And the absolute priority of questions emerged as the driving force in the act of discovery and learning. (Together with the recognition that my troubles with undergraduate mathematics had stemmed from spending so much time reading other people's answers to questions which neither they nor I had formulated.)

Some five years before, I had attended a conference in which G. Pólya had been invited to give a plenary lecture. He did not give a lecture, but he gave the participants a collection of ten quotations about the teaching of mathematics. All were striking, but one in particular has stuck in my mind ever since.

> “The object of mathematical rigour is to sanction and legitimate the conquests of intuition, and there never was any other object for it.

Pólya attributed this quotation to J. Hadamard. That quotation neatly summarised my experience of research, and became a way of understanding the task of a teacher: first to feed the intuition, and then to legitimate. The standard exposition with its ‘definition, theorem, proof’ sequence focuses on the legitimation. “It seems impossible” said Lakatos et al. (1976, p. 142) “that anyone should ever have guessed them [the theorems and definitions]”. “Yet all these matters must at one time have been goals of an urgent quest, answers to burning questions, at the time, namely, when they were created” wrote Toeplitz (1963, p. xi). At the same conference at which Pólya's ten quotations were circulated, Freudenthal, another plenary lecturer said: “Fundamental definitions do not arise at the start but at the end of the exploration, because in order to define a thing you must know what it is and what it is good for” (Freudenthal, 1973, p. 107). So a learning sequence may be different from a logical sequence.

A third insight that had come my way at an ATM conference a few years earlier was from W. M. Brooks of Southampton. He urged us as teachers of mathematics to keep a record, in a notebook, of the actual processes by which we resolved a difficulty for ourselves and came to understanding. The point of the record is that without it, our understanding is so satisfying that we forget the difficulty which we had, and thereby lose contact with the thought processes of our struggling students. A life time of becoming familiar with the concepts which students have, is our lot as teachers, as we try to do better.

So much for my background: an advanced research student expressed willingness to correspond with me, and I went to his office for an afternoon of discussion. He gave me, straight off, a summary of what an undergraduate number theory course might consist of, and an indication of different directions in which a central theme might be explored. I made avid notes in my rather small notebook but at the time understood very little of what he said. He recommended a couple of books. Then I had a sabbatical and worked through his books and parts of others, filling up my notebook with the actual computations which I had needed to do to understand what I was reading about.

It was the patterns emerging from computations which made theorems believable, and it was computations which gave definitions meaning, and, much more subtly, it was computations which showed how proofs were going to work. The giants of number theory—Fermat, Euler, Gauss—had done masses of computation and that work is a whole lot easier today with handheld calculators, or when necessary, computers.

The sabbatical referred to above had been preceded by the sabbatical of a colleague, Stuart Plunkett, who had used part of his time exploring the possible ways in which information about numbers might be displayed. He came up with just seven ways and invited us to suggest what he had missed. This may sound a somewhat limiting discovery, but it focused my attention on those forms, especially two dimensional forms, which were rarely found in the number theory books I was using, but which exposed some numerical patterns much more readily than their conventional algebraic format.

In the event, the notes I made as I worked out the meaning of theorems became the basis of *A Pathway into Number Theory*. The 800 or so questions of which the book consists pose the computational exercises which I found myself doing as I sought for understanding. Theorems too emerged as the answers to questions. In retrospect the scheme respected the Chinese proverb which was cited repeatedly during the 1960s:

> “ I hear and I forget.
> I see and I understand.
> I do and I remember.

Pólya’s quotations and Freudenthal’s lecture are to be found in Howson (1973) and the problem-solving course which was the outcome of this experience was published as *A Pathway into Number Theory* (Burn, 1982).

Chapter 5

Teaching Problem-solving Explicitly

Matthew Badger, Trevor Hawkes and Chris Sangwin

In Chapter 2 we stressed the importance of problem-solving in developing students' mathematical abilities, and discussed some approaches to helping them to master it. In Chapter 3 John Mason reflected on the work of George Pólya and how it influenced his own. In this chapter we will look at the practicalities of running a problem-solving module at undergraduate level, beginning with an overview of current practice in England and Wales.

In Chapter 2 we said what we mean by problem-solving, what we believe students stand to gain from it, and what constitutes a good problem in a given context. We now describe practical ways of getting problem-solving into the curriculum, looking closely at three particular approaches:

1. Challenging problems in the assignments, coursework and assessment of a standard module.
2. A core module entirely taught and learnt through problem-solving (known as problem-based learning or PBL).
3. A module focused on problem-solving as a valuable mathematical skill for its own sake.

We will suggest effective ways to put these approaches into practice and discuss their advantages and disadvantages. The first approach is the most commonly used and easiest to implement; the second requires a large initial investment of time and pedagogical ingenuity, but is practised at Warwick for example (see Section 7.6 below); several imaginative examples of the third approach can be found in the remaining case-studies in Chapter 7. We begin with a survey of current problem-solving practice in university mathematics departments in England and Wales.

5.1 Current Practice in England and Wales

To determine how departments of mathematical science build problem-solving into their degree programmes, we sent a short questionnaire via email to the 59 heads of departments that currently offer a Mathematics BSc degree in England and Wales[1]. From the responses, we invited six departments with a module in their degree programmes covering explicit problem-solving to be documented in a case-study. This entailed semi-structured interviews with the members of staff responsible for their problem-solving modules; the interviews that we recorded and transcribed are the basis of the six case-studies that appear in Chapter 7.

In this section we summarise the questionnaire responses and describe briefly the outcomes of the case-studies.

5.1.1 The Problem-Solving Questionnaire

The questionnaire was kept as brief as possible to encourage a large number of responses, and so the number of questions asked was strictly limited. Because interpretations vary, it began with the following description of problem-solving as:

> “. . . any substantial task or activity that calls for original, lateral or creative thinking by students; brings several ideas or techniques together in a surprising way; introduces something new; illuminates some topic, e.g. with a helpful counterexample.

This was followed by five questions on problem-solving in the respondent’s department:

1. Does your institution offer a module in any of its Mathematics Degree Programmes which requires students to engage in problem-solving?
   - No
   - Yes, an optional module
   - Yes, a compulsory module
2. What is the module code and title (choose the best example in your programme)?
3. Is problem-solving the central aim of the module?
4. What year is the module taken in?
5. Please give a brief description of the module you offer.

We asked for of respondents’ department and contact details for our records, and whether we could contact them for follow-up information with a view to choosing case-study candidates.

[1] The remit of the National HE STEM Programme extends only to England and Wales.

Summary of Replies to Questionnaire

Of the 59 institutions we approached, 34 completed the questionnaire. The responses generally indicated an absence of formal problem-solving in the mathematics degrees of English and Welsh HEIs. Twelve of the respondents acknowledged outright that they had no problem-solving in either compulsory or optional modules, even though, in half of these cases, problem-solving was mentioned in the published aims of the programme.

Of the remaining 22 respondents, 5 offered an optional problem-solving module to students, and 17 stated that problem-solving was included in one or more compulsory modules. However, further investigation suggested their interpretation of problem-solving was different from ours: sometimes it was a general claim that problem-solving was an inherent part of any mathematics degree programme, even if not tied to any particular module; sometimes problem-solving was taken to mean mathematical modelling or optimisation, even numerical analysis or statistics. After taking into consideration answers to other questions, we identified just six modules, either compulsory or optional, that could be deemed to include a substantial amount of problem-solving in our sense.

Here we should stress that, where we were at odds with respondents' interpretations of the level of problem-solving in their programmes, it was mainly because we were interested in specific modules where problem-solving formed a large part of the students' work. We freely concede that it is unlikely that any student could make it through an entire mathematics degree course without encountering *some* problem-solving of the type we have in mind. Given the importance attached to this activity, however, it is surprising that so few departments offer problem-solving as an explicit part of their mathematics degree programmes.

5.1.2 The Six Case Studies

From the questionnaire we identified six institutions offering a mathematics module with problem-solving at its heart, who subsequently agreed to be part of our case-studies. Here we give a brief overview of the case-studies, but make reference to them throughout the remainder of this chapter. They can be found in their entirety in Chapter 7.

University of Birmingham Problem-solving is taught to first-year students in their first-term using the Moore Method. There are no lectures, and the only materials students are given are the problems they are required to solve and definitions they might need to solve them. No examples or model answers are available to students. There are currently two groups using questions on different topics – geometry and set theory. Students do not get to choose which group they are in. They meet for two hours per week to present and discuss their solutions. The module is optional for BSc students but now expected for students enrolled on the 4-year Mathematics MSci programme.

Durham University Delivered in the first-term of the first-year, Durham's compulsory module, entitled *Problem-Solving*, is based on *Thinking Mathematically* (Mason et al., 2010). It is taught using both lectures and problems classes, in which students work through problems in groups, recording the progress they are making by using *rubrics*.

University of Leicester Leicester's compulsory second-year module, *Investigations in Mathematics*, sees students working in groups of around a dozen on different mathematical topics at an appropriate level. It is taught using a modified Moore Method by several members of staff working in parallel.

University of Manchester The *Mathematics Workshop* is a first-year, first-term module again using *Thinking Mathematically*, though to a lesser degree than at Durham. Students begin the year working in computer labs, before concentrating on modelling and problem-solving after the mid-term break. A compulsory module, the *Workshop* has both lectures and classes.

Queen Mary The only third-year module we studied, Queen Mary's *Mathematical Problem-Solving* is an optional module taken by a dozen students each year. Each student has a different set of problems from a range of topics in pure mathematics. Although there are no lectures and only a single class each week, students may seek help from members of staff at other times.

Warwick Warwick's first-year *Analysis 1* module has been taught using problem-solving for the past 15 years. This unique example among our case-studies is a core module taken by all its 320 first-year mathematics students, taught by problem-based learning. Students have one lecture and four hours of group work in problems classes each week.

5.2 Approaches to Teaching Problem-Solving

In this section we discuss practical ways of getting problem-solving into the curriculum of a mathematics degree, concentrating on modules designed with problem-solving as the primary activity. In Section 5.2.1 we make a distinction between teaching problem-solving for its own sake, and using it as a medium for students to learn and understand the material on the syllabus of a given module. In both cases, the aim is to use problem-solving to engage students in a mathematical apprenticeship, and to develop their ability to 'think mathematically'.

In Section 5.3 we describe ways of giving problem-solving a higher profile in the existing modules of an established degree programme, with minimal additional investment. Not every mathematics department will have the resources to develop new problem-solving modules, which our case-studies show to require—at least initially—a large investment of time and energy by staff. Introducing students to problem-solving is, wherever possible, a worthwhile activity, and so this section demonstrates the ways in which this may be done.

While our suggestions and recommendations below are informed by the literature, by our case-studies, and by personal experience, we do not claim that these are the only ways to teach problem-solving successfully. We urge our readers to experiment with new approaches and to monitor students' performance and feedback to improve the experience.

5.2.1 Teaching Problem-solving and Teaching *with* Problem-solving

Teaching problem-solving is distinct from teaching *with* problem-solving, in both aims and approach, though we will argue that the outcomes are frequently similar. Teaching problem-solving concentrates not on the mathematical content of what students are learning, but the ways in which they approach unseen problems in any area of mathematics. In two of our case-studies, those at Durham and Manchester, students are taught to solve problems by employing Mason's *rubrics*, in which the process of problem-solving is reflected upon, carefully documented, and its stages identified. When these techniques have been understood and absorbed, students can apply them elsewhere.

The case-study modules at Birmingham, Leicester and Queen Mary teach problem-solving by giving students problems to solve, without explicit guidance on the solving process. Here the topics the problems cover is of secondary importance; for example, students in Leicester and Birmingham are divided into groups covering completely different areas of mathematics, while those at Queen Mary answer problems from a range of topics in pure mathematics.

Unique among our case-study institutions, Warwick uses problem-solving to teach the core first-term, first-year module *Analysis 1*. Instead of a traditional lecture course (three hours of contact a week in a large lecture theatre), the students have a 1-hour lecture and two 2-hour classes a week. In these classes, students work at tables in small groups through a series of ten workbooks, that contain the definitions and problem sequences that form the bulk of the material they need to learn. Students from other departments who take *Analysis 1* in their first-year are taught by the standard lecture method, and do not perform as well on the final examination as their peers in mathematics (Alcock and Simpson, 2001).

Warwick began teaching analysis by problem-solving in 1997, after a pilot study in 1996. The change required considerable commitment from a number of members of the department, the investment of more resources than other modules, and more postgraduate assistants.

While the benefits of this approach are clear, it is understandable that the teaching of a core module entirely through problem-solving is unlikely to gain ground, especially when taught to a large cohort so early in the programme. In contrast, creating a module dedicated to problem-solving as a worthwhile skill in its own right is perhaps a more realistic aim.

5.2.2 Classes

Problem-based learning does not prescribe that all interactions with students should be in a tutorial or group-study environment. In four of our six case-studies, lectures—being a more efficient method of imparting information—are used to some degree to supplement classes, where students spend the majority of their time working on problems alone or in groups.

In problem-solving modules at Durham, Manchester and Warwick, classes proceed in similar ways; students work in groups on solutions to the problems they are given. Staff and teaching assistants spend time with the individual groups to encourage and cajole them, and to assist in various ways: perhaps offering hints when they are stuck, or checking solutions before students present them at the board. Working in groups has both advantages and drawbacks for students – mathematicians rarely work in isolation and so group work reflects the nature of mathematics; however students that are, or perceive themselves to be, struggling may be tempted to depend on others to do most of the work. Students quickly identify the strongest in a group and risk devaluing their learning by becoming passive members. In the Section 5.2.4 we discuss how potential problems can be avoided through assessment.

At Birmingham, Leicester and Queen Mary the practice is somewhat different. At Queen Mary, students work on disjoint sets of problems and so there is no collaboration at all. They have a single, one-hour class each week during which time they can get assistance from the teacher; they are also encouraged to see staff outside of the class, an opportunity which almost all take advantage of. In classes at Birmingham and Leicester, one student presents a solution at the board while others offer their comments and criticisms, and in this way work as a group. At Birmingham, students are supposed to work on problems alone outside of classes, however, collaboration does sometimes occur and students are often honest about who contributed most to a given solution. At Leicester, students are encouraged to work together outside classes, and have a number of group assessments (a poster and presentation), for which collaboration is required.

The problem-solving modules at Durham, Leicester, Manchester, and Warwick are compulsory; those at Birmingham and Queen Mary are optional to students on the 3-year degree. The modules at Durham, Manchester, and Warwick use lectures for some of their teaching. Here we come upon the issue of resources: Leicester has the fewest students with around 100 a year, while the largest, Warwick, has as many as 320 to cater for. They need far more teachers, rooms, time-table slots, and administration than a standard lecture module, and to help reduce the teaching burden supplementary lectures are used at these three institutions. Table 5.1 outlines the use of resources in our six case-study modules, and clearly demonstrates that a compulsory module taught using problem-based learning in a large department imposes a considerable burden on the organiser.

Module	Classes	Class Size	Staff	Assistants	Class hours	Lecture hours
Birmingham	2	14	2	0	2.5[1]	0
Durham	7	18	7	0	2	1
Leicester	9	10	9	0	1	0
Manchester	9	30	9	9	2	0.5[2]
Queen Mary	1	12	1	0	1	0
Warwick	11	30	11	11	4	1

Table 5.1: Division of labour in six problem-solving modules

It is understandable that a department may be reluctant to dedicate such resources to a single module, though the benefits are clear where they have been measured at Birmingham (Badger, 2012) and Warwick (Alcock and Simpson, 2001). It is interesting to note that both these departments expanded their provision for problem-solving teaching; at Warwick a pilot study in 1996 took place before the new module was adopted for all first-year mathematicians, while at Birmingham the Moore Method module that was initially optional for all first-year students is now expected of those on the MSci programme.

Levels of Assistance

The level and type of help offered to students solving problems strongly influences the difficulty of the problems they will be able to tackle and the speed with which they answer them. At one end of the continuum is a traditional lecture approach, in which a technique or procedure is demonstrated by the teacher, and where students need only to emulate the method to correctly complete the exercises. At the other end of this same continuum is the Moore Method, in which teachers give the barest of help to students, typically responding to a question with another probing question instead of giving an instruction. Most problem-solving teaching sits somewhere nearer the Moore end of the spectrum—lest problems turn into

[1]This represents two and three hours of classes, alternating weekly.
[2]This represents a single, one-hour lecture a fortnight.

procedural exercises—but that is not to say that giving students specific guidance on occasion has no pedagogic value.

Lecturers who begin teaching problem-solving for the first time are frequently surprised by the extent to which students struggle with seemingly straightforward problems, especially near the start of a module. The *didactic contract* (Brousseau, 1997) between student and teacher must be clearly articulated at the start of a problem-solving module; students who do not appreciate the point of such a module may be tempted to solve problems through means other than their own cognitive exertions. The overwhelming experience of the staff interviewed for our case-studies is that students who understand the aims of their modules recognise the importance of working for themselves, and resist the temptation to short-circuit the sometimes painful struggle to solve a problem. Nevertheless, lecturers may prefer to select their problems from an obscure source (such as a text-book in a foreign language) to minimise the risk of cheating and to reassure colleagues.

The most important aspect of teaching students to solve problems on their own is to put them in a position where relevant mathematical thinking occurs spontaneously. To give them directions or to promote a particular heuristic is counterproductive – students need to be given time to be stuck and to find their own ways out. The principle job of the teacher is to help students to discover their own natural mathematical powers and bring them to bear on challenging problems – a deft touch with minimal intervention is required to achieve this.

5.2.3 Topics and Tasks

Having discussed the roles of the teacher and the student in the problem-solving process, we now focus on the third most important aspect, namely the problems themselves and the tasks and skills required to solve them.

By accident of birth, our Project is aimed specifically at problem-solving in the context of pure mathematics. (Another project, also funded by the National HE STEM Programme, is aimed at mathematical modelling and applied problem-solving.) For this reason, five of our six case-studies are in pure mathematics, the exception being Manchester, which is also involved with the modelling project. We freely acknowledge the central importance of mathematical modelling and believe it to be an indispensable part of any mathematics degree. No student should graduate without knowing how to translate an amenable real-world problem into a mathematical form, and to learn solve it, to test its validity as a model, to refine it and to analyse its limitations. We focus on the 'solve it' stage as an indispensable part of the modelling process. Moreover, almost all topics taught in a typical first degree in mathematics—pure or applied—are amenable to a problem-solving approach

When setting up a problem-solving module for the first time, we strongly advise against trying to author original problems, not least because of the amount of work required. Writing tailor-made problems that are of sufficiently high-quality to give students the best experience can be left until the module has run once or twice. Warwick, for example, began teaching analysis using Burn's *Steps into Analysis* (1992) before original workbooks were written in its second year of operation. Birmingham's geometry class used Gutenmakher and Vasilyev's *Lines and Curves* (2004) for several years before Dr. Sangwin wrote new problems covering Apollonian circles, and Leicester's graph theory class used Moore-Method resources from the *Journal of Inquiry-Based Learning in Mathematics*. After a problem-solving module is established, it is advisable to review the problems in the light of student and staff feedback and student performance, and to consider writing a fresh set, or at least revising the existing problems, to improve their effectiveness and suitability. If there is a risk of students colluding from one year to the next, having several sets of problems can help to avoid it.

A further recommendation is to restrict the content of the problems in a module to a particular area or topic in mathematics. In this respect, our case-study modules vary their approaches; in Birmingham, Leicester and (naturally) Warwick, problems form a progression that—taken as a whole—develop deeper knowledge of a particular subject area. At Manchester, students cover a range of topics that last two weeks each, while at Durham and Queen Mary problems are entirely disconnected and drawn from different subjects in pure mathematics. We have a number of reasons for preferring a more focused approach:

1. Students become familiar with a substantial body of related mathematical knowledge.

2. Students learn how mathematical ideas develop, evolve, inter-relate and contribute to a coherent theory.

3. With a deeper understanding of a particular topic, students are more likely to play with ideas, speculate and to make conjectures of their own.

Although problem-based learning does not usually allow as much material to be covered as conventional lectures (Warwick's analysis module has two more hours of contact time than a standard module each week), there is evidence that it leads students to have a firmer understanding of the mathematics and to outperform those who learn the same material by traditional methods (Alcock and Simpson, 2001, p. 106). If the material covered in a problem-solving module is to be a prerequisite for other modules, it is important to underestimate what students will achieve in the allotted time and to limit the formal content of the syllabus.

While getting students to experiment with concepts and formulate their own conjectures does not seem to be a feature of the case-study modules, it is certainly an important part of what it means to be a mathematician. In *Thinking Mathematically*, Mason et al. (2010, p. 40) emphasise the importance of students learning to specialise and generalise the problems they are given and make them their own:

> “ Mathematical thinking does not begin until you are engaged in a question. The most engaging question is always your own, either because you made it so by specializing and Entry activity, or because it arose from your experience.

It would be natural to teach a topic that is strongly related to one's research – one is more likely to know a range of interesting problems not already documented in the literature. But this is not necessary for success; indeed there are potential benefits in choosing an area of mathematics with which one is less familiar. R. P. Burn's highly-regarded *Numbers and Functions* (1992) was produced as a result of his struggles with understanding analysis; here his lack of expertise made it easier to see where students may face difficulty, and to develop those areas with care and clarity (see Chapter 4 for Bob's account of this process). Along similar lines, Dr. Walker in the Manchester case-study is a group theorist who teaches topics from applied mathematics while author C.J.S. at Birmingham is an applied mathematician teaching geometry.

The key considerations when choosing a topic for teaching problem-solving are whether it will engage students and teaching staff alike, and whether there are suitable problems available with which to get started. The first is a matter of personal judgement, and for the second we direct readers to Section 5.4, and to this project's website, for sources for good problems in a range of mathematical topics.

Documenting the process of problem-solving

As an important part of the learning process, students a Durham and Manchester are introduced to the idea of *rubric writing*, as defined by Mason et al. in *Thinking Mathematically*, to document systematically their own experiences of solving problems:

> “ If you did not write things down, then you missed an opportunity to learn something about yourself and about the nature of thinking. I recommend that you take the time to work through most questions conscientiously. . . . It does at first seem awkward and unnecessary, but a little self-discipline at this stage will reap rewards later. . . .
>
> The framework consists of a number of key words. As you use these words they become endowed with associations with past thinking experience, and through

> these associations they can remind you of strategies that worked in the past. . . . The whole framework of key words is called a RUBRIC, following the medieval custom of writing key words in red in the margins of important books. The activity of writing yourself notes I call RUBRIC writing. – Mason et al., 2010, pp. 15–16

Both Durham and Manchester encourage students to document carefully the process by which they solve problems; at Durham students are also marked on their rubric writing. Such rubrics divide the process of problem-solving into three phases, *Entry, Attack* and *Review*, and serve to prompt students to remember what worked for them in the past.

Whether or not one chooses to teach students directly from *Thinking Mathematically*, there are certainly ideas within the book that will inform anyone's teaching of problem-solving, and we recommend it highly. Rather than rehearse its ideas here, we now move on to discuss other tasks and practical issues that are helpful for teaching problem-solving.

Using Class-Time Effectively

When it comes to solving problems, students need encouragement, emotional support, and, in particular, to feel they are not struggling alone. Though classes are the best place for a lecturer to give reassurance and set the tone, there are other helpful ways of using the time in class, and our case-studies again reflect a range of approaches. At Durham students working in groups present their solutions to problems during workshops, while at Birmingham and Leicester students are expected to solve problems in their own time, and most of class time is spent by students presenting their solutions at the board. Frequently these presentations are more a process of collaborative iteration, where students present partial or potential solutions that their peers then critique, suggesting improvements or clarifications. This, it is argued (Coppin et al., 2009), gives students further experience of what it means to be a involved in a mathematical seminar, though some students find it particularly difficult, especially during the early stages of the module.

5.2.4 Assessment

The traditional end-of-module timed examination can be a good medium for testing a student's understanding of a body of mathematical knowledge. It is less effective, however, for assessing a student's problem-solving ability. It is much harder for an examinee to have the right idea come to mind, to try out different plans of attack, to cope with getting stuck, and so on, with the clock ticking away in the claustrophobic atmosphere of an examination hall. Even a talented problem-solver can perform poorly under these conditions. (That said, young people do learn to succeed in such conditions, for instance in Mathematical Olympiads and in the Sixth Term Examination Papers (STEP) set by Cambridge Assessment for applicants to over-subscribed degree courses.) Warwick uses a traditional timed exam as partial assessment for its PBL module *Analysis 1*, but more to test syllabus knowledge than problem-solving ability. Manchester has a class test at the end of the module worth only 10% of the final credit. Birmingham and Durham both previously used final examinations, but both were dropped in preference for ongoing assessment. For these reasons we do not recommend the use of timed exams as the main form of summative assessment for problem-solving modules.

Continuous assessment may create more work for students and teachers during term-time, but it reduces the burden on both come exam time. Furthermore, ongoing assessment may encourage less conscientious students to keep up with the work more so than standard formative tasks do. With its emphasis on skills rather than knowledge, problem-solving is best assessed at the stage at which skills are being learnt and practised. Furthermore, whereas a final examination is normally just summative, ongoing assessment has the added benefit of also being *formative*, about which Black and Wiliam (1998) had the following to say:

> “ The research reported here shows conclusively that formative assessment does improve learning. The gains in achievement appear to be quite considerable, and as noted earlier, amongst the largest ever reported for educational interventions.

Each of the six modules in our case-studies, apart from that at Warwick, use the submitted solutions to problems as a large part of the module assessment and their experience is a strong recommendation for this approach. At Birmingham and Queen Mary, straightforward solutions to problems make up 50% and 100%, respectively, of the module credit. At Durham 25% is awarded for a problem solution, and a further 50% for a project including an extended problem. At Leicester students’ workbooks, including solutions to problems, are worth 50% of the module mark, while at Manchester project reports are worth 60% total, of which 40% comes directly from problem solutions. At Warwick, students submit solutions to problems on a weekly basis, and these are marked but not used for assessment.

Other forms of assessment that are used by our case-study modules and which may prove worthwhile for teaching problem-solving include presentations and computer-based assessments. The module with the largest presentation component is *Developing Mathematical Reasoning* at Birmingham, where students receive 50% of their module marks for their best two presentations of solutions to problems at the board. At Leicester, students in groups give presentations on the topics that they have been working on for 20% of their final mark.

Assessing Group Work

Working in groups allows students to develop their skills at collaboration and generate new ideas faster, however some students may become reliant on others to solve problems on their behalf. To avoid this issue, we recommend that group assessment should not account for more than a small amount of the total marks available for a module. At the four universities that include group work in their problem-solving activities (Durham, Leicester, Manchester and Warwick), those at Durham and Warwick have no group component to their mark, while Leicester and Manchester have only 30% (from a poster and presentation) and 10% (from completed projects' group average) respectively.

5.2.5 Summary

So far in the chapter we have looked at ways of designing modules that give students a significant experience of mathematical problem-solving. Our six case-studies indicate a variety of successful approaches, and we will now summarise the lessons of their experience with a check list of things to bear in mind when setting up a problem-solving module. The key areas are teaching, topics and tasks, and assessment.

Teaching

- During the module, most of the student's time should be devoted to solving problems, in class and private study, working alone or in groups. Lectures and class discussion have a part to play too.

- The teacher must exercise careful judgement about the level of challenge each student is ready to cope with; knowing the students' mathematical backgrounds and present capabilities greatly helps when exercising this judgement.

- Students learning to solve problems that are challenging for them need considerable support and encouragement. They must be allowed to be stuck but not left alone to the point of becoming demoralised.

- What level of support is appropriate must be gauged on an individual basis. Any help offered should be incremental, and can include: checking that the student has understood the formulation of the problem: encouraging the student to ask questions; responding to a question with another question; directing a student's attention to earlier problems; suggesting further reading; and, as a last resort, giving small hints.

- When students have solved a problem, they should be encouraged to write up a polished and rigorous solution. Presenting their solution to their peers for comment and suggestions for improvement is a salutary exercise for all parties.

- Students benefit from reflecting on their problem-solving and keeping records of the thought-processes, strategies and techniques they used to reach a correct solution. This reviewing activity can be formalised by requiring students to write rubrics that help them to build up a personal repertoire of successful approaches.

- Questioning, the process of specialising and generalising, modifying or extending an existing problem, is an important part of being a mathematician; with encouragement, students can learn to do this themselves.

- Computer software such as *GeoGebra* is valuable for visualising the problem, doing hard calculations, exploring various approaches, studying special cases, checking conjectures, and so on.

- Setting up a problem-solving module calls for considerable time, effort, and commitment from the module organiser. Delivering it can also be more demanding than a standard lecture course.

Topics and Tasks

- The problems handed out should be commensurate with each student's abilities, presenting them with sufficient challenge while offering a fair chance of success within a reasonable time.

- Choosing problems from a specific area of mathematics has advantages: the students learn the techniques that work in that area and acquire a deeper understanding of that particular body of knowledge. There is also something to be said for exposing students to problems that bring together ideas for different areas so that they come to appreciate the inter-relatedness and essential unity of mathematics.

- There is value in creating sequences of problems, where each problem builds on those before and the progression leads to a significant result at the end. Students become familiar with the circle of ideas and techniques involved and derive satisfaction from the sense of having undertaken a significant mathematical journey.

- Creating a body of suitable problems from scratch is hard. When setting up a problem-solving module, it is advisable to borrow tried-and-tested problems from a variety of published sources. Only when the module has become established should the leader be tempted to write original problems to suit their style, the subject matter, and the students' backgrounds

Assessment

- Timed exams are better for assessing knowledge than problem-solving skills, which function more effectively in a relaxed study environment.
- Continuous assessment can evaluate the various skills in action and also give students an incentive to attend the classes.
- Course credit can be offered for a variety of activities: original solutions, presentations, rubrics, portfolios, fair copies of solutions, posters and other types of record.
- Credit can be awarded for group work, but carries the risk of weaker students becoming free riders.

General

- Teaching problem-solving is important and effective. It is not a substitution for learning mathematics but a key component of the process.
- Setting up and running a problem-solving module calls for a significant, but worthwhile, investment of resources. Beginning with an optional module on a modest scale will enable a department to analyse its benefits before making a full-scale commitment.

5.3 Integrating Problem-solving in an Existing Course

In this section, we will suggest three different ways of introducing problem-solving into a standard lecture-based mathematics module, and we begin with some hypotheses about its structure. There are exceptions to the generic model we propose, but it is fair to say that most students are taught mathematics this way in U.K. universities.

The bulk of the material that students have to learn is presented in the module lectures, which typically last just under an hour and are timetabled two or three times a week. Assignments in the form of exercise sheets are handed out regularly, say every fortnight, and are the only formative assessment a student will have. They are sometimes followed up with hints or full solutions once students have submitted their answers. Conscientious students work through assignments as a module proceeds and, by staying abreast of the material in this way, are able to keep up with lectures and have a better chance of understanding new material as it is presented.

Regular tutorials (also known as ‘examples classes’) support students as they learn the module material and work on the assignment questions. These tutorials are optional and run by teaching assistants or postgraduate students who sometimes have little or no training in teaching. The teaching assistants and postgraduates collect students’ submissions, apply a marking scheme set by the lecturer, add suitable feedback, and return them to the students. For each of these assessments a small amount of credit is awarded, perhaps up to 4% of the module mark for each submission and 20% as a whole. Students later use these returned assessments when revising for a final examination, the main assessment for the module.

The questions on the assignments are mainly exercises, building predictably on subject matter or worked examples covered in the lectures. They may be organised in sequences leading to a useful result that there was no time to include in the lectures. These assignments frequently include a number of optional ‘starred’ questions, designed to challenge the more capable students. While these questions are closer to problem-solving in our sense, they are usually not marked and only a few of the more ambitious or conscientious students will even attempt them. By including challenging questions in this way, a module may contain the potential for problem-solving; however, students have no incentive and most perceive no need to attempt them.

The practical approaches we describe in the next three subsections aim to engage more students in problem-solving within the confines of a generic module. The scenarios are based on the authors’ and colleagues’ experiences and have been tried with some success. They depend crucially on the role of the module leader, who ideally should

- believe in the value problem-solving
- have the time and enthusiasm to promote it
- have the resources to support and encourage it
- have a module structure for regular assessment, feedback, and credit.

The ideas can be mixed and matched to suit local conditions.

5.3.1 “At least one of these problems will be on the final exam”

This announcement is made at the start of the module and repeated from time to time. In our hypothetical, but hopefully typical, 10-week module with fortnightly assessments, each assessment contains two or three substantive starred problems; the ‘problems’ referred to in the section title. Promising that one or more of the starred assignment questions will appear verbatim on the final exam does not, by itself, have much impact. The more ambitious students will work on the problems and probably share their solutions with friends, and by exam-time the most organised students will have learned these solutions by heart. There are, however, better ways of getting more students more closely engaged with the problems and their solutions:

1. Design problems that are central to the module syllabus so that by doing them students reinforce their understanding of the core material and are made fully aware of this benefit.

2. Prepare the way for tackling a starred assignment problem with a sequence of exercises that provide the necessary tools and insights for solving it, thus solving the problem becomes the reward for working through the exercises and joining the ideas together.

3. Teach to the problems. Talk them up in the lectures, highlight their importance and show how they fit into the lectured material. Attach distinction and prestige to the solving of these problems.

4. Make the solving of the starred problems a central part of the tutorials/examples classes and emphasise their importance to those running the classes. If the classes are run by teaching assistants or research students, train them to mediate problem-solving in class without giving away the solutions. Group work on problems is often a good way to achieve this.

5.3.2 Rewarding Students for Solving Problems

The main idea here is to devote more of the module contact time to problem-solving and to reward this activity with module credit. Here is one such scenario that can be applied to a first- or second-year module:

1. While preparing the lectures, look for (say) ten significant ideas that are closely related to the module material (a corollary to the theorem, a change of hypotheses, a counter-example), or a result would have been included if there were more time. Work them up into problems or, even better, a problem sequence.

2. Formulate the problems in a way that makes it harder to find their solutions online, even inventing some new terminology if necessary (for instance, author C.J.S. used the term ‘circumflip’ for reflections of circles in lines).

3. Divide the cohort into classes of 18 or 24 students each and timetable a weekly slot (even one of the lecture hours) for the problem session; find enough separate rooms for the classes to meet and collaborate (a seminar room with tables to work at is ideal).

4. Recruit enough volunteers from the year above, preferably mathematics students who have previously taken the module in question, and train them as peer tutors, working in pairs, to organise and moderate the problem-solving sessions. Peer tutors role are called ‘student leader’ at Loughborough, where author T.O.H. witnessed very successful peer tutoring as part of their National HE STEM project *Second-Year Maths Beyond Lectures* (SYMBoL); we will adopt this term too.

5. The unseen ‘Problem of the Week’ is handed out at the start of the session and the class divided into three or four groups of six, working together at a table. Within each group, they work in pairs for around 10 minutes, unpacking the problem, thinking about its meaning. They are then asked come together with their group to compare notes and try to come up with a strategy for solving the problem. Each group appoints a scribe who keeps record of the discussion and ideas. They continue to work on the problem for another 15–20 minutes (roughly the half-way stage), again splitting up to share the labour if they so choose.

6. As the groups are working, the two student leaders circulate among them and observe their progress, making discreet enquiries to clarify their understanding of what has been happening. At the half-way stage, groups that feel they are badly stuck may opt to accept a pre-arranged hint handed out by a student leader. They then work for another 15 minutes, with or without the hint, at which point they are called to order and asked to focus on writing up their work during the last 10 minutes.

7. At the end of the session, each group hands in its written account of their work. The students leaders add their own observations of each groups’ performance and write a preliminary qualitative judgment of the submitted work and perhaps a suggested mark out of 10 that takes into account whether or not the hint was used. The module organiser later moderates the assessment by the student leaders and decides on a final mark for each group in each class.

8. The SYMBoL Project at Loughborough highlighted the importance of careful training of the student leaders to give them the confidence to manage the sessions with authority (but a light touch), and to provide uniform treatment to all groups and classes. It is vital that they know how to mediate the learning process without giving direct help or hints to the groups.

9. There is a good case for specifying the membership of the groups in advance of the sessions and permuting the members each week; this evens out the variations in mathematical and problem-solving ability and encourages a spirit of cooperation within the class. As a variation of the above assessment structure, the groups can be given the opportunity to continue to work on the week's problem outside class and to submit a final draft of their solution by the start of the next session. Module credit can be given for the considered solutions but, of course, this involves extra marking.

10. The value and importance placed on the problem-solving activities should be reflected in the level of module credit awarded. Students should know at the outset that their individual mark coincides each week with the mark awarded to the group they are in. With group work, author T.O.H. has experimented with offering groups the option to decide how to divide the marks among themselves, but they usually felt this was invidious and did not take it up.

5.3.3 A Low-Budget Large-Scale Approach

The following method was used for a number of years in the Mathematics Department at the University of Warwick, for a first-year number theory module taken by 150 students and also for a third-year representation theory module taken by 60. Both topics were suited to multiple-choice testing, because the subject matter offered scope to use calculations to measure a student's understanding and simultaneously to provide plausible distractors.

We will concentrate on the number theory module, which ran for five weeks of the summer term, immediately before the start of the exam period. The syllabus material was delivered entirely in five weekly workbooks, written by Warwick staff in the style of books by Burn (1992; 1997), mostly concentrating on sequences of problems. Topics covered included modular arithmetic, arithmetic functions, inversion formulae and continued fractions; many of the problems included calculations.

The workbooks were distributed (online and in hard copy) at the start of each week, when students sat a 50-minute multiple-choice test on the content of the previous week's workbook. The test was held in the lecture theatre allocated for the module and came in four versions, with questions permuted to minimise plagiarism. Test questions were based on the exercises and problems in the workbooks, and answer sheets were marked automatically by optical reader and processed by computer. Scores were normalised so that random guessing led to a zero score on average; negative scores were raised to zero. Two sets of tests were created for use in alternate years to minimise the chance of students passing down the answers from year to year. Students were handed complete answer sets as they walked out of the test and the solutions to the workbook questions were then published online.

Each test accounted for 5% of the module credit, and so a quarter of the final marks were based on multiple-choice questions. There was a close correlation between the average test scores and performances on the final exam, which also contained workbook problems. University regulations prevented greater weight being given to the tests as they were classified as coursework. From year to year, statistical analysis of the test results helped to increase the proportion of discriminating questions and to improve the multiple-choice distractors.

The module always began with a single organisational meeting to tell students how it worked. In the early years, weekly support classes were also provided, but they were later replaced by self-moderated forums, where students could discuss issues as they arose in their workbooks. (Warwick offered an excellent range of home-grown online learning tools, like the forums, which, in particular, let users embed mathematics in web forms using basic LaTeX markup.)

The development of the resources for such a module (the workbooks, the multiple-choice tests, and so on) calls for a very significant commitment, and staff involved in the preparations will be much encouraged if their departmental teaching or administrative load is realistically reduced to compensate. In the Warwick instance, once established, the number theory module took very little time to organise and there were few administrative overheads after the initial investment. There were no lectures or tutorials and the only contact was the hour for the weekly test, which could be run by a competent teaching assistant.To weigh up, here are the pros and cons:

Pros Low maintenance; the module runs itself; the students enjoy it; scales up well; good student engagement.

Cons Heavy up-front investment; approach better suited to some topics than others; general suspicion of multiple-choice testing (although this author thinks it can be more searching than is commonly realised); more routine exercises than genuine problems in the workbooks.

For more information, questions or comments, please email trevor.hawkes@coventry.ac.uk.

5.4 Sources of Good Problems

There are a great many sources of problems in print and online that can form the basis of a problem-solving course. This section pretends neither to be comprehensive nor complete, though the sources we recommend have been used by one or more of the authors in some capacity.

We tentatively offer the set of problems available on our project website, though currently a work-in-progress at the time of this publication: www.mathcentre.ac.uk/problemsolving.

5.4.1 General Problem-solving

Books	
How to Solve It Pólya (1945)	Póya's was the original text on teaching problem-solving, and is still a set text in many mathematics education degree programmes. The problem-solving modules at Durham and Manchester both make use of *How to Solve It*.
Mathematics and Plausible Reasoning Pólya (1954a,b)	While *How to Solve It* gave a précis of Póya's theories of teaching problem-solving, in the two volume *Mathematics and Plausible Reasoning* he expanded greatly on the previous work. Both volumes draw on extensive examples from many areas of mathematics.
Mathematical Thinking and Problem-solving Mason et al. (2010)	A comprehensive book on the current state of problem-solving teaching and what it means to think mathematically. Also used at Durham and Manchester, *Mathematical Thinking and Problem-solving* sees many of Polya's ideas updated, as described in Chapter 3.
The Art and Craft of Problem-solving Zeitz (1999)	With less exposition than *Mathematics and Plausible Reasoning*, Zeitz's book contains a large number of interesting problems from a wide range of mathematical topics.
Solving Mathematical Problems Tao (2006)	Accessible to A-Level students, Tao's book uses a wide range of problems from pure mathematics to challenge readers.
Techniques of Problem-Solving Krantz (1997)	Something of a classic, with a very large number of problems from a number of topics in mathematics, a solutions manual is also available (Fernández and Gooransarab, 1997).

5.4.2 Puzzles and Recreational Mathematics

In addition to these books, the British Mathematical Olympiad website (www.bmoc.maths.org) contains a number of links to problems used in past instances of the Olympiad, and in some cases solutions are included. Notably the complete problems (PDF link) from 1993-2011 are available for download.

Books	
Martin Gardner	It is difficult to overstate the contribution that Gardner made to the practical job of collating interesting problems and puzzles. *My Best Mathematical and Logic Puzzles* and *Entertaining Mathematical Puzzles* are just two examples of his work.
The Moscow Puzzles	Written by Boris Kordemsky and edited by Gardner, the Moscow Puzzles are a highly regarded set of puzzles from a range of mathematical disciplines.
Puzzle-based learning Michalewicz and Michalewicz (2008)	Michalewicz's book expands a great deal on the ideas outlined in Section 2.1.3, and contains a large number of puzzles.
The Red book of Mathematical Problems Williams and Hardy (1988)	The problems are especially chosen for students preparing for undergraduate math competitions, but these challenging brainteasers will be of interest to anyone interested in math problems dealing with real numbers, differential equations, integrals, polynomials, sets, and other mathematical topics.

5.4.3 Areas of Mathematics

The *Journal of Inquiry-Based Learning in Mathematics* (JIBLM) has an extensive collection of course notes on its website for use in PBL courses which cover analysis, calculus, foundations, geometry, graph theory, group theory, number theory, statistics, topology and trigonometry.

Books	
Lines and Curves Gutenmakher et al. (2004)	Subtitled *A Practical Geometry Handbook*, Lines and Curves was used at Birmingham when the module MSM1Y first introduced geometry.
Numbers and Functions: Steps into Analysis Burn (1992)	Bob Burn's *Steps into Analysis* was used as the original basis of the analysis module at the University of Warwick. His number theory and group theory books are also highly recommended.

Chapter 6

Problem-solving and Computer-aided Learning

Sue Pope

The Problem-Solving in Undergraduate Mathematics (PSUM) project is a sister project of the *Mathematics Problem-Solving Project*, also supported by the MSOR Network and funded by the National HE STEM Programme. In this chapter Sue Pope, the project leader, puts forward the case for integrating computers into the problem-solving process, and demonstrates the *interactivities* that she and her team worked on with students.

The central aim of the project was to develop an innovative and sustainable online bank of starting points for problem-solving. These would be presented in an interactive, visual, and engaging way to nurture mathematical thinking, logical processes, and modelling. The starting points permit a range of teaching approaches – individual, small group, and whole class. Designed to be used on a range of computers and mobile devices, the four starting points will be available on the NRICH website from September 2012.

6.1 Computer-Aided Learning

Accurate calculation has always been an important aspect of mathematical activity, leading to the development of machines that compute quickly and reliably by Pascal in the seventeenth century, Babbage in the nineteenth century and Turing and his successors in the twentieth century. As affordable computing power has become increasingly available, the range of ways in which calculations can be performed and the outcomes presented has enabled exploration of multiple possibilities that could not be realised practically due to prohibitive expense and/or the amount of time involved.

It is hard to imagine a context in modern society where computers would not be used to collect and analyse data, to model complex systems (such as the weather or disease), and to explore possible outcomes when solving problems. Recognising the strengths and limitations of technological tools is essential to making effective use of them. Users need to be able to decide whether or not to use technology and select the most appropriate technological tool. With the rise of mobile devices students can work on tasks and problems wherever they may be. The nature of the technological tools makes a significance difference to the quality of learning (Hosein, 2009).

In 1980 Papert set out a vision for how computers might transform the way in which mathematics is learnt. Logo was a computer language that allowed children to ask their own questions, teach the computer, and learn from feedback. Almost 20 years ago Becta (1993) declared that technology allows learners to explore mathematics in a way that enables them to:

- Learn from feedback: The computer provides fast and reliable feedback that is non-judgemental and impartial. This can encourage students to make their own conjectures and to test out and modify their ideas.
- Observe patterns: The speed of computers and calculators enables students to produce many examples when exploring mathematical problems. This supports their observation of patterns and the making and justifying of generalisations.
- See connections: The computer enables formulae, tables of numbers and graphs to be linked readily. Changing one representation and seeing changes in the others helps students to understand the connections between them.
- Work with dynamic images: Students can use computers to manipulate diagrams dynamically. This encourages them to visualise the geometry as they generate their own mental images.
- Explore data: Computers enable students to work with real data that can be represented in a variety of ways. This supports interpretation and analysis.
- 'Teach' the computer: When students design an algorithm to make a computer achieve a particular result, they are compelled to express their commands unambiguously and in the correct order; they make their thinking explicit as they refine their ideas.

Research by Pierce et al. (2007) suggests that students' mathematical skills and confidence to tackle problems is loosely associated with a willingness to use technology. Research into the use of computer algebra systems suggests that students can become expert users undertaking mathematical exploration with understanding (Pierce and Stacey, 2001; Coupland, 2004). The challenge for lecturers is how to incorporate the use of technology in a way that nurtures mathematical learning and empowers students to have ownership of the tools to solve problems.

6.2 PSUM

The *Problem-Solving in Undergraduate Mathematics* project produced interactive starting points, named *interactivities*, aimed at developing students' mathematical problem-solving at undergraduate level. The *interactivities* were carefully specified and designed by the team and then implemented by an experienced mathematician-programmer with a view to being available on a range of technological platforms. All were trialled with undergraduates or other students working on undergraduate level mathematics (e.g. graduates on a subject knowledge enhancement course prior to embarking on a PGCE), and all students and tutors completed anonymous online surveys which provided evidence of efficacy. Some students participated in focus group discussions on the individual *interactivities*, with members of the team.

Preliminary data were collected for all of the interactive starting points, while in the early stages of development. The survey respondents were predominantly male (68%), with an average age of 28. The majority of responses were mature students on a Mathematics Subject Enhancement course at Liverpool Hope University. Although most of the respondents were not undergraduates, they did have equivalent mathematical qualifications to the average undergraduate student i.e. A-Level Mathematics, with about one in five having completed A-Level Further Mathematics. A somewhat surprising finding was that students predominantly chose to solve the problems using traditional pen and paper methods, with software applications (such as Autograph) being far more popular than calculators. More people chose to work with others than individually (see Figure 6.2.1).

The large majority of student participants found that they obtained a good or very good understanding of the mathematical concepts contained within each problem-solving scenario and 97.4% of student participants found that the simulations were helpful for learning mathematics.

The students' responses to the interactive starting points were overwhelmingly positive. They found the problems attractive, interesting, enjoyable, and easy to use. When solving the problems students found that the opportunity to interact, and the visual presentation of the problems, encouraged them to explore the context thoroughly and so they gained insights that helped to solve the problems. Students were keen to explore more problems presented in this way.

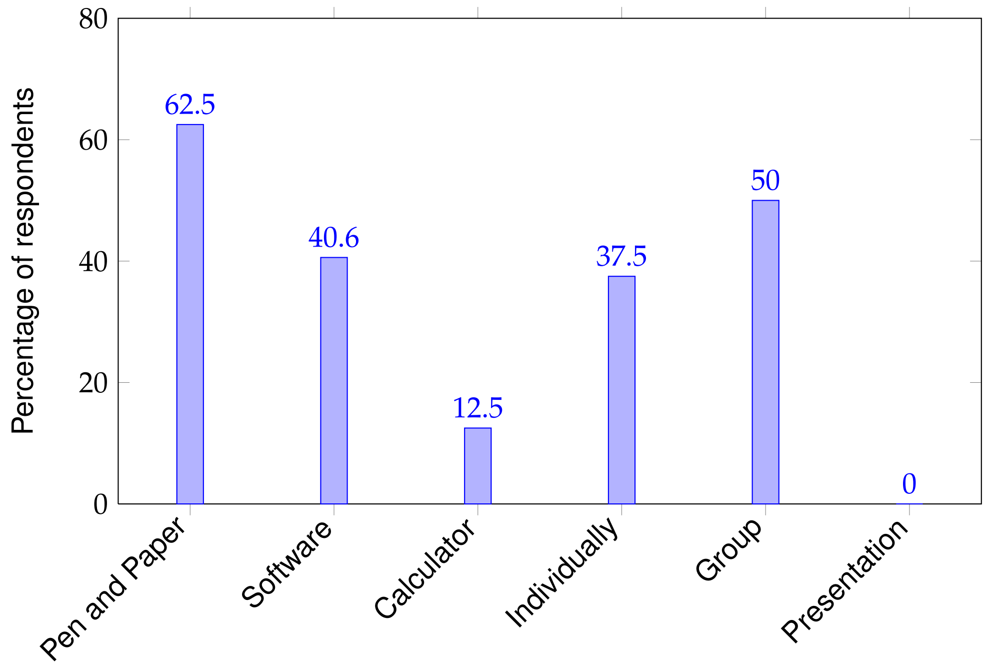


Figure 6.2.1: How did you choose to work on the problem?

6.3 *Interactivities*

This section covers the *interactivities* on which the students worked, and a brief discussion of their comments from the online surveys and discussion groups.

6.3.1 Picture This!

'Picture this!' was used with a group of five first-year Cambridge undergraduates, who tried the activity at the end of their second-term. They had learned about Euclid's Algorithm during their first term, and so it was hoped that this would revise some ideas and give them another way to visualise them (however Euclid's Algorithm was not specifically mentioned to the students at any time). The interactive picture (Figure 6.3.1) was displayed to students using a projector and accompanied by the questions give in Example 6.3.1.

▼ Example 6.3.1 — *Picture This! Questions*

Here are some questions for you to consider:

- *What are the important features? What is allowed to change and what must stay the same?*

- *Given two positive integers, can you produce the corresponding diagram (without the computer)? Compare your diagram with the one obtained from the interactivity.*
- *Do different pairs of integers necessarily lead to different diagrams?*
- *Can you start from a diagram and work back to the initial numbers?*
- *Can you produce some diagrams that are somehow 'extreme' examples?*
- *Can you describe one of these diagrams in words and/or equations?*
- *What might the diagram illustrate? You might wish to make some conjectures that you can test (and, if necessary, refine). Try to prove your conjectures.*

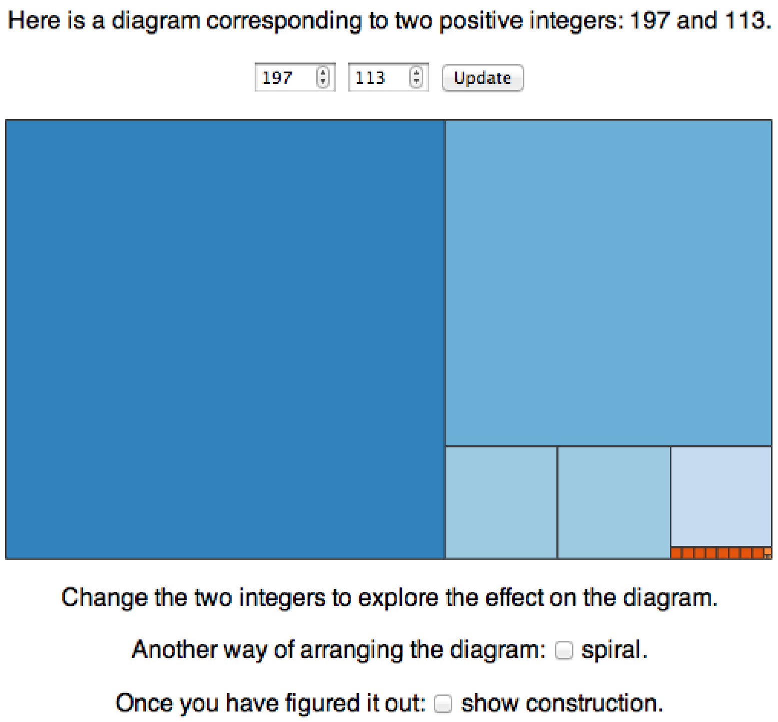


Figure 6.3.1: The *Picture This* web interactivity

Students were encouraged to think about the questions, and to change the numbers to gain an idea of what was going on. While initially not keen to interact with the picture (which would require them to go to the front of the class), students spontaneously entered into a discussion of the problem for the next 40 minutes. Students alternated between discussing things together, and working alone.

The qualitative feedback captures how the computer delivery supported problem-solving. A clear theme was the benefit of instantaneous feedback: “[The resource was] Interactive, quick to generate answers”, “you could easily change the starting numbers, and gaining an accurate visual representation is useful”, “you could try different integers and see how they affected the diagram”, “Quick visual results to confirm findings”, and “[The interactivity], enabled me to check my answers quickly.”

Respondents generally reported deepening and reinforcing their knowledge of mathematics. “[I] understood the concept of golden ratio prior to using it, however it made it easy to illustrate how it works using remainders”, “[The] visual [aspect, helped me] to understand the concept, [it also] made it easier to see where you’d made any mistakes”, “It helped to reinforce Euclid’s algorithm”, “[It] provided an alternative way of explanation”. More generally students commented that, the interactivity was “very pleasing on the eye and user-friendly”. However, one student reported “[I] wasn’t quite sure what I was meant to be doing, even after we had finished. . . ”; something a tutor may need to take account of.

The last ten minutes were spent discussing with the tutor what the students had found, while bringing some of their ideas together. The students had clearly identified that the picture linked to Euclid’s algorithm, and could describe articulately how it did so. Most seemed to like having another way to visualise Euclid’s algorithm; they had thought about which numbers require many iterations and which require few, although they hadn’t developed this all the way to proofs (e.g. that consecutive Fibonacci numbers are ‘worst’). Overall, *Picture This!* worked well to develop students’ understanding, even in a relatively short space of time.

6.3.2 Graphs

The ‘Graphs’ *interactivity* was used with a group of twenty students on a Mathematics Enhancement Course at Liverpool Hope University. The Mathematics Enhancement Course comprises several five-week modules covering various topics within the A-Level and undergraduate mathematics curricula; the course is designed to provide non-specialist graduates with sufficient mathematical background to become secondary school mathematics teachers. Used in the second lesson of the *Decision and Discrete* module, the *interactivity* allows students to add and remove vertices and edges on a graph, and to rearrange its vertices. Besides a few who had encountered graph theory during A-Levels, the majority of students were meeting it for the first time. The session was held in a computer laboratory in order to allow individual use of the *interactivity*; a PowerPoint presentation was used to guide the session. Students were first introduced to the definitions of the features of graphs including vertex, edge, region, and degree of a vertex, before being given access to the *interactivity* itself.

Students explored together how to add/remove a new edge and vertex using the built-in functionality. Initially, some students found it difficult to connect one vertex to another but these difficulties were soon overcome. Their first task was to explore the relationship between the sum of the degrees of the vertices and the number of edges, which they did by drawing several simple graphs, and using them to develop conjectures. They then created increasingly complex graphs and tested the conjectures with them.

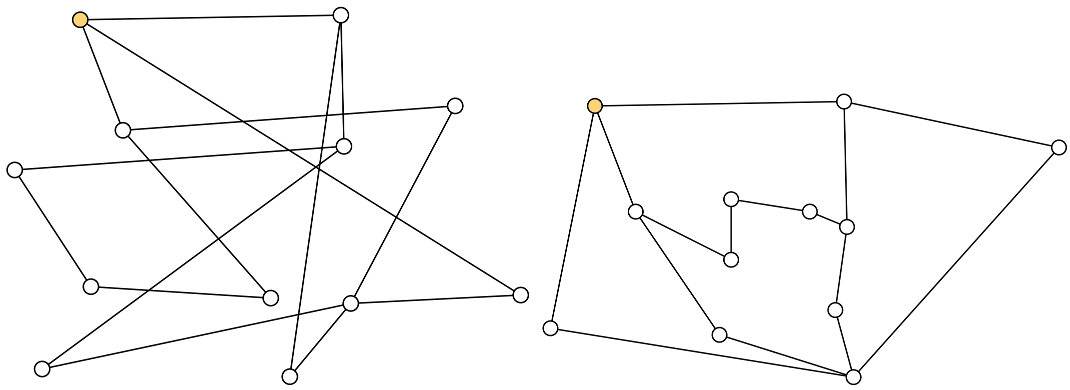


Figure 6.3.2: Unjumbling planar graphs

After exploring different types of graphs, students were encouraged to find the relationship between the number of vertices and number of edges in a tree. Most students adopted a similar strategy, drawing simple trees with fewer than ten nodes before increasing the number of nodes in order to test their conjectures. After introducing the notion of a complete graph, students explored how many edges there are in a complete graph with n vertices. Students started simply with a triangle and moved up to more complex figures such as an octagon.

Students also explored planar and isomorphic graphs using the interactivity, as demonstrated in Figure 6.3.2. They enjoyed doing this particular task, because they liked creating complex graphs and investigating how to manipulate them to become planar.

This *interactivity* provided an accessible introduction to graphs and allowed students to develop understanding of important concepts through exploration. All respondents found this starting point helpful for learning the mathematics and exploring the problem. Qualitative responses from the students focused on the visual, interactive environment as “easy to manipulate”, and “nice visual representation”, which allowed students to “experiment with different arrangements quickly”. The students also confirmed that using technology has distinct advantages over paper based experimentation because, “(I can) move nodes around”, “(it is) easier than drawing”, and “[it allows] seeing where you are going and have been”.

6.3.3 Linear Programming

The 'Linear Programming' *interactivity* was also used with students on the LHU Mathematics Enhancement Course, in the first lesson of the *Decision and Discrete* module where most students were meeting linear programming for the first time. The session was again held in a computer laboratory with a PowerPoint presentation used for guidance. Students used the mathematical software *Autograph* to solve the simultaneous equations $y = 2x + 5$ and $2y + x = 8$, before determining the linear inequalities that defined the various regions on the same graph.

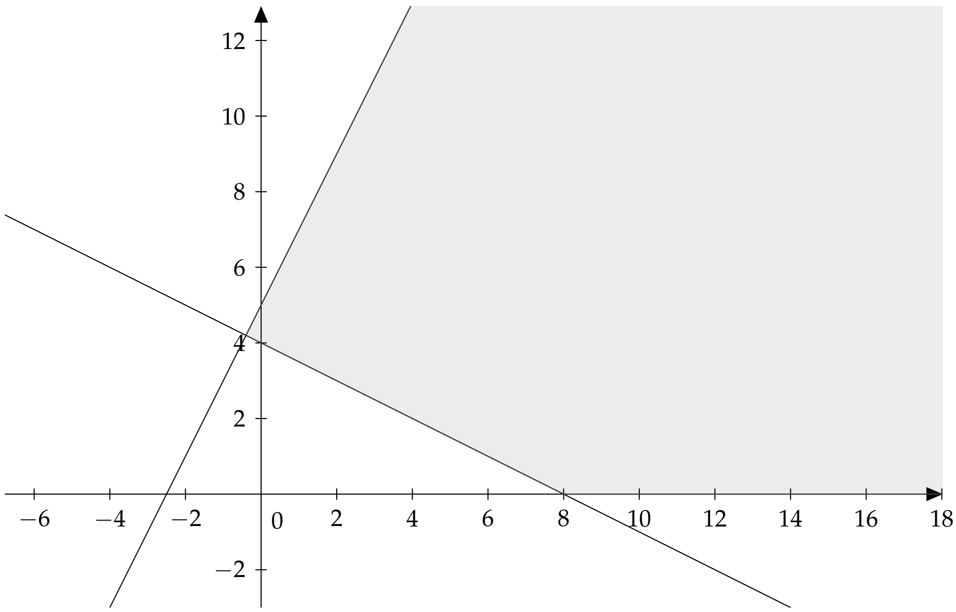


Figure 6.3.3: The region defined by the inequalities $y \geq 0$, $2y + x \geq 8$, $y \leq 2x + 5$

Once students were confident with this prerequisite knowledge they were introduced to the linear programming *interactivity* in a web browser. This consisted of the interactive graph (Figure 6.3.4) and question in Example 6.3.2:

▼ Example 6.3.2 — *Toys*

Woods, Inc. is a toy manufacturer. Currently it manufactures wooden dolls and wooden trains. The current number of carpentry hours available per day at Woods, Inc. is 100 hours. Dolls usually take about 1 hour of carpentry whilst trains take about 2 hours. The time it takes to paint these toys is similar, i.e. 1 hour each, but there are only 80 painting hours available per day. Wooden trains are not the biggest seller and hence Woods, Inc. tries to keep the manufacturing of wooden trains to be 40 or less per day. Currently, dolls sell for £1 *whilst the wooden trains sell for £2. Let the number of trains be x and the number of wooden toys be y. Given these constraints how can Woods, Inc. maximise profit?*

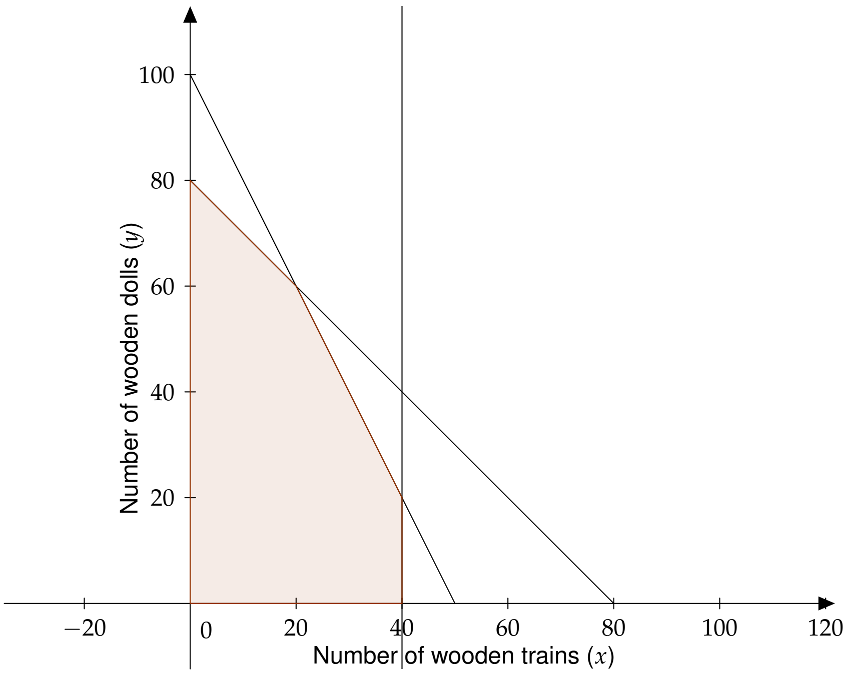


Figure 6.3.4: Graph to solve Example 6.3.2

Below the *interactivity* was a list of questions for the students to answer. The first six covered basic questions such as the maximum of individual products' production given unlimited funds. The last three questions were as follows:

7. What is the minimum number of dolls and trains that can be produced? Why?
   (a) Where is this point on the graph located?
8. Given the constraints of painting, carpentry hours and demand, what is the maximum:
   (a) Number of dolls that can be produced? Why this number?
   (b) Number of trains that can be produced? Why this number?
9. Play around with the production of dolls and trains whilst having a look at the graph until you find the best profit.
   (a) At what point on the graph can you find the best profit?
   (b) Is it at an intersection of lines?
   (c) Why can you not find the best profit above the intersection of the lines?

Students worked through the first six questions and used their answers to plot constraint lines. By interacting with the plotted graph, they answered Questions 7 and 8 and two additional questions:

- What is the profit for the maximum number of dolls produced?
- What is the profit for the maximum number of trains produced?

They then tackled Question 9 in the simulation to optimise the profit. Students were encouraged to vary the original constraints and investigate how the feasible region changed. They learned how to solve the linear programming problem by using a sliding rule to represent the profit line, or by evaluating the profit for extreme points of the feasible region.

Students were quick to identify the feasible region, describing it as the "area where the constraints all apply at the same time". They were pleased to see that the feasible region could be displayed automatically.

Some students struggled with Question 7, and needed to be reminded about the practical context:

> “ **Tutor.** *Think in real-life now, if I had a toy factory and I was producing the dolls and trains, what is the minimum number of toys can I produce?*
> **Student.** *Zero.*
> **Tutor.** *Good! And where will that point be on the graph?*
> **Student.** *At the origin!*

The students then worked on optimising the profit. When asked where the best profit occurred some students noticed,

> “ **Student.** *It is where the lines intersect.*
> **Tutor.** *Do you think it is always where the constraint lines intersect?*
> **Student.** *Yes, I think so.*
> **Tutor.** *How do you know which intersection is correct? Why don't you change your constraints and see if that is always the case?*

The students then experimented by varying the constraints and optimising the profit in each case. Many could justify the approach of using a sliding rule to represent the profit line or by evaluating the profit for extreme points of the feasible region in order to optimise the profit for any set of constraints.

Overall, the session went well and proved more effective than with other approaches previous used to teach the same material on the Mathematical Enhancement Course. Students commented that they found the interactivity easier to use than Autograph, and liked being able to see the feasible region. They liked being able to drag the constraint lines and explore how changing the constraints impacts the profit.

All respondents found this starting point helpful. The visual representation and interactivity were warmly received: “It allowed quick changes to be made, and allowed a visual freedom I’ve never experienced before in linear modelling”, “Simple to use. Being able to alter in real time is very helpful”. Nearly all felt that it helped them learn the relevant mathematics, because “[it] makes you feel like you’re doing more” and “[I] like to see visual demonstrations of theory”. Other students reported how the use of technology facilitated the problem-solving process, e.g. “It allowed quick updates to the solution space, which in turned allowed me to determine if I’ve made silly mistakes, and hence improve my intuition as a maths student”.

More general responses included: “I can see great use of this software in a teaching environment to get the idea of what is going on across faster. . . I found it refreshing to see something presented so simply. It was extremely good”.

6.3.4 Filling Objects

The ‘Filling Objects’ *interactivity* was the last of the three used with the Mathematics Enhancement Course at LHU. The interactivity was introduced during the final session of a twelve session calculus module, whose focus had been predominantly pure calculus techniques. The session was again held in a computer laboratory, though this time there was no PowerPoint presentation. Little guidance other than “explore the environment” and “gain a greater insight into the situation using mathematics” was given to students.

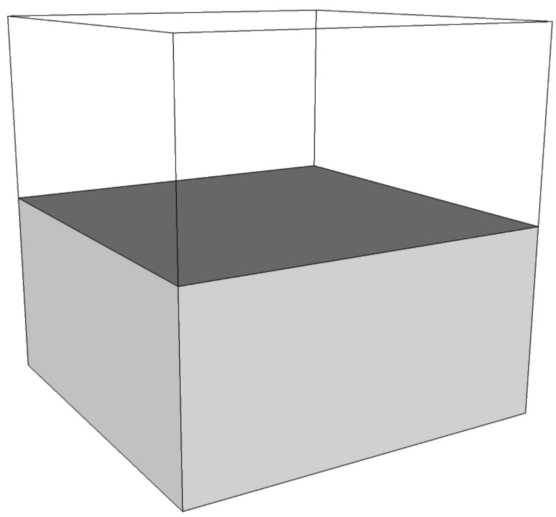


Figure 6.3.5: Partially filled cube

▼ Example 6.3.3 — *Filling cone*

Water is flowing into a conical vessel at rate x. The vessel has the shape of a circular cone with horizontal base, the vertex pointing downwards; the radius of the base is a, the altitude of the cone b. Find the rate at which the surface is rising when the depth of the water is y.

In this interactivity, a three-dimensional shape was shown to fill with water. Figure 6.3.5 shows a cube; other shapes displayed were a sphere and a cylinder; the parameters of radius/side length, input rate, and shape type varied every time the webpage was reloaded. Students were required to use what they learned from the three shapes to determine what would happen in the case of a circular cone, as given in the question (Example 6.3.3). Below the question was a input box into which students could type an answer.

Students were initially quite lost, since problems of this kind had not been explicitly introduced in the module thus far; they began discussing the question between themselves and developed an understanding of what was being asked. Although not all the students were able to attempt the mathematics during the early stages, none of the students ever said they were stuck. The students were quite happy to continue exploring the different shapes and generating questions. One student managed to receive the notification of being correct using the cube and other students gathered around to explore the solution. Following some discussion the other students went away and achieved the same 'Correct' notification on their own work. At this point students began to work more collaboratively, sharing ideas and approaches. When one student shared his correct solution other students didn't understand why he was correct. Despite his explanations, some were unconvinced.

Students enjoyed the session even though they did not all achieve the same level of understanding, and many found the experience challenging. They found the Filling Objects *interactivity* to be a useful tool for exploring and trying out their answers, "without needing the teacher".

Responses were more mixed to this *interactivity*. Most found it useful, but only one respondent reported achieving a good understanding of the mathematics by the end of a two hour session. This may be linked to the demands of the mathematical content associated with this *interactivity*. Less than half the respondents reported that the starting point was helpful for learning mathematics, however most found the simulation useful for exploring the problem and appreciating the visualisation: "the 3D illustration gives you a clue how to tackle the problem", "watching the rate change helps to visualise the problem at various stages", and "the visual effect was great".

6.4 Outcomes and the Future

The four interactive starting points for problem-solving work on various technological platforms, including mobile devices, and permit a range of teaching approaches. They took longer than anticipated to develop and this needs to be borne in mind when undertaking any future developments. Students and tutors completed anonymous online surveys which provided evidence of efficacy and informed the online guidance which accompanies the online resources. The analysis of student responses is included in the section above. There were too few tutor responses to make any meaningful analysis possible. Respondents found all of the starting points useful for engaging with mathematical concepts through problem-solving. They encouraged active exploration of the problem exemplifying many of the Becta (1993) benefits: learning from feedback, working with dynamic images, and exploring data.

The starting points will be available on the NRICH website, nrich.org.uk. All four were developed using open-source software and are themselves released under an open-source license, so they can be developed further and new starting points created in future.

Chapter 7

Case-studies

Matthew Badger

The following case-studies form part of the basis for the advice we offer in this guide. As standalone documents, the material they contain will overlap in places with that found in Chapter 5. These case-studies were created from transcripts of interviews with the staff at the six institutions; we are most grateful for their time and assistance.

7.1 University of Birmingham

Developing Mathematical Reasoning is a first-year module in the School of Mathematics at the University of Birmingham. The aim of the module is to develop students' problem-solving skills and raise their confidence when engaged in doing mathematics. Started by Dr. Chris Good in 2004/05 (Good, 2006), in 2007 1Y was taken over by Dr. Chris Sangwin. Classes are limited to a maximum of 20 students per group, and in some years two sections have run in parallel with each using different problem sets. The module is advertised to students in the module handbook as follows:

Module	Developing Mathematical Reasoning
Year	1st
Code	MSM1Y
Hours	2 or 3 hours alternating weekly
Length	One term (11 weeks)
Mandate	Expected for MSci students
First run	2004/05

> MSM1Y is a *Moore Method* course. The Moore Method is a version of the Socratic method pioneered by the Texan mathematician R. L. Moore. The aim is to learn a subject by exploring it for yourself rather than being taught (i.e. 'told') it.
>
> 1. No books or other reference material!

2. Everything you write down should be written in English and make grammatical sense as English, even if it is not correct mathematically.

The aims of MSM1Y are to develop

1. an ability to approach and solve problems independently;
2. a logical writing style adopting appropriate conventions; and
3. mathematical confidence.

Teaching is ostensibly divided into 'lectures' and 'example classes', though there is no distinction between types of contact time in the module. There are two 'lectures' a week, with 'examples classes' being held on alternating weeks; in some years the timetable results in a double slot on these weeks.

For purely administrative reasons, the module forms half of an optional twenty credit *Module Outside the Main Discipline* (MOMD). Students may select any MOMD from a wide range throughout the university, although many elect to choose a MOMD offered by their own school. From 2011 onwards students enrolled on the four year M.Sci. programme were expected, but not required, to choose this module and were guaranteed a place on it. In cases where students wish to take a foreign language for their MOMD they are encouraged to do so, even if on the M.Sci. programme.

In 2011/2012 two separate groups took *Developing Mathematical Reasoning*, completing problems in either naïve set theory (taught by Dr. Good) or plane geometry (taught by Dr. Sangwin).

Teaching

Developing Mathematical Reasoning is taught using the Moore Method, named after the influential Texan topologist Robert Lee Moore (1882–1974), see Parker (2005), who developed it for his university mathematics courses. The Method proceeds in the following manner:

1. Mathematical problems are posed by the teacher to the whole class.
2. Students solve the problems independently of each other.
3. Students present their solutions on the board to the rest of the class.
4. Students discuss solutions to decide whether they are correct and complete.
5. Students submit written solutions for assessment.

The module uses a close approximation of the Moore Method. In class the teacher 'chairs the seminar', ensuring that the discussion remains at a professional level and on task, makes careful notes of any solutions offered, and assigns marks for presentations. The teacher's key role is encouraging students in their work, both at the board and during discussion. Students are called by the teacher, in turn, to answer problems; later in the term the strict order is relaxed to ensure that all students have a fair chance to contribute. In some instances students are quietly and privately warned they will be asked to present a particular solution, but in general every student should expect to be called upon to present their solution to any problem.

When asked to present a solution to a problem, a student may decline. If so, they will be asked to answer another problem later; it is always at the discretion of the teacher who gets to answer which problems, however. Solutions to problems are expected to be of a similar standard to that of a member of staff in a normal lecture. Once a solution has been presented the rest of the group may ask questions to clarify an issue or point out an error. If a error is discovered that cannot be resolved there and then, the student at the board may opt to work on the problem further, or another student may take over. This decision is always made by the teacher, however if the student at the board wants to revise their solution this will always be accepted.

Once a student has presented a solution to a problem, they or anyone else may offer an alternative solution or propose some extension of it. The class progression is strictly linear; only once a problem has been solved may a solution to the next be offered. In this regard the teacher acts as a 'safety-net', and will point out an error only if the rest of the class agrees on an incorrect solution to a problem. They never provide worked solutions to problems.

Problems, and the axioms and definitions required to solve them, are the only information given to students. The problems aim form a coherent sequence within a substantial piece of mathematics. This is one key defining feature of a Moore Method course.

Dr. Good chose problems from naïve set theory. A short list of definitions and axioms, together with all problems, were distributed at the start of the course. Dr. Sangwin chose problems from plane geometry. Originally taking many problems from Gutenmakher et al. (2004), the problems have evolved significantly over the last five years. In the first class students are given a list of 'facts from school geometry', to which students may appeal without justification. These play the role of axioms, without starting at the lowest level of Euclid's *Elements*, and include Pythagoras' and Thales' Theorems. During the first few classes students each present a 'fact', in turn, at the board; illustrating them with diagrams or GeoGebra worksheets. This allows students to get used to presenting without the added pressure of defending their ideas to the class. Problem sheets follow each week with between two and eight problems depending on the progress made. Problem sheets occasionally introduce further definitions where required.

Group Work

Students are encouraged to answer the problems on their own, without use of reference materials, however classes contain a strong element of group discussion. This structure gives two phases to the process: private work and public discussion. In classes errors or inefficiencies in a proof will be pointed out by other students, and so a student may redraft their solution for a later class, taking the other students' comments into account. In this way collaboration does occur. In practice students do collaborate outside class, and are often honest about who 'got the idea' when asked. The overwhelming experience is that students appreciate the need for them to solve problems for themselves, and plagiarism has never presented an issue in the module.

Assessment

From 2011 there is no final exam in 1Y; assessment is solely on students' presentations at the board and their written solutions to all the problems.

1. Each student's best two **presentations** are assessed by the teacher, based on their completeness and clarity. Students are not penalised for a solution which is not the shortest or most concise, but will be marked down if the contents of the board do not offer a complete solution. Preference is given to correct solutions, although this is not absolutely required. Presentation marks count for 50% of a student's final mark.

2. The **written solutions** that students submit to problems account for the remaining 50% of the marks for the module, which works as follows:

   - Students submit solutions to problems, on a per-worksheet basis. These are usually submitted once a week, but there is a lag of at least a fortnight between solving the problems in class and submitting solutions (students are encouraged to submit early, however).
   - The teacher marks the students' work. Each solution is marked as either incorrect or correct; there are no half marks. Where a solution is incorrect, the teacher will indicate on a student's submission where and why a particular solution fails, though without direction to a possible answer.
   - Each worksheet's solutions may be returned to a student once, if there are mistakes, and the student can then make appropriate changes to improve their mark.

- The second submission of a worksheet's solutions is final, but there is no penalty for solutions correct on the second submission.

Motivations and Development

The motivation for the module came from the belief among many colleagues in the School of Mathematics that, while students coming from A-Level had strong procedural skills and content knowledge, they were unable to tackle unseen problems to a sufficient level. Further, students had difficulty in forming and expressing cogent mathematical arguments, especially when it came to writing these arguments down. The module was devised to prompt students to think mathematically, and for themselves, and not to teach a particular area of mathematics in a polished and professional format. In particular, the three central aims are to develop:

1. An ability to approach and solve problems independently.

2. A logical writing style adopting appropriate conventions.

3. Mathematical confidence.

By enabling students to solve problems independently they develop a deeper understanding of the material they are learning, and mathematics is transformed from a set of facts to be waded through into a discipline recursively constructed. In turn, by building such a view of mathematics in the minds of undergraduates they have ownership of the material they are learning.

The Moore Method came to Dr. Good's attention through colleagues in the United States, and while Moore himself used the method only with graduate students, in the U.S. it is widely used at first-year undergraduate level and beyond. The method had not been previously used at Birmingham, nor by any of the staff who have subsequently taught the module, though it has changed very little since its inception in 2004/05. When it first began the module was optional, and taken by between 12 and 15 students each year, with two to four dropping out after they had started. Originally too, departmental regulations required an exam, counting for 50% of the mark, to be sat by students in the summer. This exam was a mixture of problems taken directly from the module, and new problems developed from those previously seen by students.

Problem Selection

Since it began in 2004/05 the module has used four problem sets:

1. Set theory problems from 2004/05 were written by Dr. Good.

2. Geometry problems from 2007/07 used *Lines and Curves: A Practical Geometry Handbook* (Gutenmakher et al., 2004).
3. Set theory problems for 2011/12 were again written by Dr. Good, adapted from his originals.
4. Geometry problems for 2011/12 were written by Dr. Sangwin, based on Apollonian circles.

Central to each of these sets of problems is the progression from one problem to another allowing students to learn something of a particular topic, while also gaining an appreciation of the structure and *construction* of mathematics as a whole.

▼ Example 7.1.1 — *Cat on a Ladder*

A cat is sat in the middle of a ladder leaning against a vertical wall. The ladder slides to the floor. Describe the path of the cat.

Example 7.1.1 is the first question from the first iteration of *Developing Mathematical Reasoning* as a geometry module, and in spite of its simplicity prompts much discussion among students. One helpful aspect of the problem is the number of ways in which it may be solved, and so a student may choose to demonstrate a proof different to that already given by someone else. Only the first two chapters of *Lines and Curves* (Gutenmakher et al., 2004) were used in the module, such was the speed of progression through problems, and answers are never more complicated than a straight line or a circle, or collections of these objects.

▼ Example 7.1.2 — *Set Theory*

Remark 5. *Below we define certain operations on sets such as union, intersection, power set. We assume that collections that result by applying any of these set-theoretic operations to sets will result in another set. So, for example, if A and B are both sets then so is the union of A and B.*
Definition 6. *Let A and B be sets. We say A is a subset of B provided if $x \in A$, then $x \in B$.*
Remark 7. *We write $A \subseteq X$. Some people write $A \subset X$ instead of $A \subseteq X$.*
Problem 8. *What does it mean to say that A is not a subset of X?*

Example 7.1.2 is an extract from early in the module book for the set theory version of *Developing Mathematical Reasoning* in 2011/12; as the module progresses remarks and definitions thin out and there is a greater density of problems. As with geometry, these problems begin *very* simply for students who are among the top performers at A-Level, and yet progress is similarly slow.

Outcomes and the Future

Students progress through problems very slowly, especially near the start of the module. Discipline is required on the part of the teacher to ensure that they do not give assistance to students, besides resolving ambiguity of meaning in the problems and definitions. The majority of students rise to the challenge presented to them by MSM1Y and perform well, though a few students are never comfortable with the method. The requirement that students present on the board, and whilst doing so engage in dialogue the rest of the class, helped a lot of students to develop their confidence; frequently students who appear at the start to be lacking in confidence turn out to be the strongest problem-solvers. Students who take MSM1Y perform statistically significantly better than their peers who do not take the module, independent of starting ability (Badger, 2012). Overall, the module appears to achieve its aim of developing students' problem-solving abilities and confidence.

7.2 Durham University

Module	Problem Solving
Year	1st
Code	Math1041:Core B2
Hours	2 hours a week
Length	One term (10 weeks)
Mandate	Compulsory
First run	2006/07

Problem Solving[1] is a first-year, first-term module at Durham University's Department of Mathematical Sciences. The module has been running for five years and concentrates on teaching students to solve problems and to write proofs, and draws much inspiration from Mason et al's *Thinking Mathematically* (1982; 2010). Students spend time discussing the ideas in the book, besides solving problems themselves. Its entry in the first-year module handbook is as follows:

> “This module gives you the opportunity to engage in mathematical problem solving and to develop problem solving skills through reflecting on a set of heuristics. You will work both individually and in groups on mathematical problems, drawing out the strategies you use and comparing them with other approaches.

Teaching

Problem Solving is a compulsory module, taken by the entire first-year cohort of roughly 130 students each year. The cohort is divided arbitrarily into seven groups of around 18 students, each taught by a separate member of staff. Teaching alternates weekly between a one-hour lecture followed by a one-hour seminar, and a single, two-hour seminar. The basic materials for the module are a set of problems that students work through, both individually and in groups.

Lectures are used to draw out ideas that have been presented by students during the seminars, or in their submitted answers to the problems in the module. Lectures refer to relevant sections of *Thinking Mathematically*, and students are encouraged to read these along with working on the module's problems.

Two key ideas that students take from *Thinking Mathematically*—and on which the book has a lot to say—are the *schema* and a *rubric*. The *schema* is used to describe the process of problem-solving itself, which it divides into three phases: ‘entry’, ‘attack’, and ‘review’. These three phases bear close resemblance to Pólya's four-step process of:

1. Understand the problem.

[1] For administrative purposes, *Problem Solving* is the first half of the two-term module *Problem Solving and Dynamics*. Insofar as the Department of Mathematical Sciences is concerned, the two halves are entirely separate.

2. Make a plan.

3. Carry out the plan.

4. Looking back.

Learning which phase of the problem-solving process they are in allows students to think about what they need to do next, and what they need to achieve in the current phase before moving on. The may find themselves moving back from *attack* to *entry*, however, if an attempted approach appears not to work.

A *rubric* is the process by which students document their *own* problem-solving process, with reference to Mason et al's schema. At each phase they write down their thoughts while working on a problem, reflecting on previous problems' rubrics to prompt thinking on the current problem. For example, during the *entry* phase, students are encouraged to write "I know X", where X is something that they are given by the question, and "I want Y", where Y is something that they are looking to find. By documenting their own problem-solving via rubrics, students develop their own 'internal monitor', allowing them to make the most of their experience in problem-solving, and to give a more structured approach to later problem-solving endeavours.

The first two lectures introduce students to the schema and rubrics, while later lectures cover each phase of mathematical problem-solving in more detail. Besides discussing the process of problem-solving, lectures cover other relevant topics, such as how to write mathematics.

In the seminars that take up three-quarters of the modules' contact time, students begin by discussing a problem they will have been given earlier in the week, before moving on to new problems. New problems are discussed and students present potential solutions to them, and develop further by suggesting their own extensions of the problems. Most problems will take twenty minutes or more to be discussed, solved, and extended, though a few each year take the whole of a two-hour seminar. The point at which the group moves on to a new problem is at the discretion of the seminar leader, as appropriate to the level of interest or progress made by the class. Problems are selected for each individual seminar, enough to ensure that students will not run out, and those problems are not reached in that seminar are not used in following seminars.

The ability to write a correct mathematical proof is one of the key aims of the module. Students are required to apply what they have been taught in the lectures, and discussed in the seminars, to create proofs that are largely correct, in an appropriate format and with suitable mathematica language (including correct grammar).

Solutions to some problems are uploaded to the university's virtual learning environment (VLE), for students to review once they have submitted their own solutions during seminars. For other problems, staff will upload only their rubrics to the VLE, for students to see that the problems are not necessarily immediately solvable, even for experienced mathematicians. To help with this approach, new problems are included each year that the teaching staff do not attempt to answer before students do.

Group Work

Students are encouraged to work together in seminars, though this is not compulsory; if a student wishes to work alone they are free to do so. The majority, however, are happy to work with their peers, and in doing so learn something about the collaborative nature of mathematics.

Assessment

The mark students are awarded for the module is based on three pieces of submitted work. The first, worth 25% of the total, is the submission of a rubric that explains how they approached solving a particular problem; the choices they made and their reasons for doing so. The second piece of work, also worth 25%, is a solution to a particular problem that has to meet the standards of a substantially complete mathematical proof. The final piece of work is a project done over the Christmas holiday on an extended problem from the module; this is submitted at the beginning of the mock exam period after the Christmas break.

When the module first began the second 50% of its mark was by an examination of a subset of its problems, however because this relied on particular problems being reached by each of the different groups, it was straightforward for students to determine which problems would be on the exam and hence learn their solutions by rote. Taking this issue into consideration, the last two years have seen *Problem Solving* using the Christmas holiday project better to assess the aims of the module.

Motivations and Development

Adrian Simpson joined Durham from the University of Warwick in 2006/2007, becoming the Principal of Josephine Butler College and a Reader of Mathematics Education. He joined the university with the expressed purpose of setting up the new module. He had taught a similar module in the third-year of Warwick's mathematics degree programme for nearly a decade, and taken the same module as an undergraduate. Dr. Simpson was previously involved with the first-year analysis module that used problem-solving, described (with Lara Alcock) in (Alcock and Simpson, 2001) and in Section 7.6.

The aims of the module are to develop students' problem-solving skills, to improve their mathematical writing and their rigour in creating proofs. *Thinking Mathematically* is used to help students understand the process by which problems may be solved, and to improve their confidence when approaching unfamiliar problems. As mentioned in the previous section, the module originally included an examination in its assessment. Besides this development, and now being taught to more groups of fewer numbers, the module has remained unchanged since it began.

Problem Selection

Problems are drawn from a number of sources, notably *Thinking Mathematically*, though these have been further developed over the time that the module has been running. Other problems have come from correspondence with colleagues in the department and beyond. British Mathematical Olympiad problems are considered unsuitable for this module as the mathematical ideas involved are essentially A-Level, while *Problem Solving* is aimed at developing new thinking about proof and proof-writing.

The problems in the module are not related to one-another, though they are almost exclusively from pure mathematics, and many come from number theory in particular. Being a module in the first-term of the undergraduate programme, the prerequisites for *Problem Solving* are minimal – students are not expected to have a Further Mathematics A-Level and the questions later in the module do not rely on knowledge learned during other modules taken during the term.

▼ Example 7.2.1 — *Square Differences*

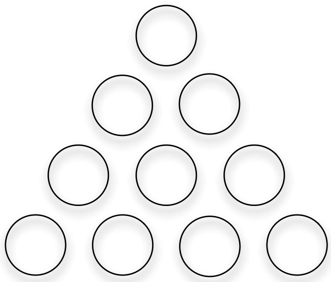
Which numbers can be written as the difference of two perfect squares? E.g.

$$6^2 - 2^2 = 32.$$

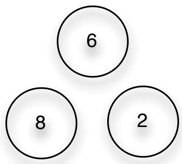
Example 7.2.1 is the type of question that would take students around 20 minutes to solve in their groups, and a little more time outside of a seminar to write a correct proof. Example 7.2.2 below shows a problem that students will be given for homework, in addition to completing the proofs of problems given in seminars. Here they would not be expected to find a general solution for the numbers 1 to n where n is a triangle number.

▼ Example 7.2.2 — *Triangle of Differences*

Consider the array below:



Can you enter the numbers 1 *to* 10 *into the array so that each number is the difference of the two below it, e.g.*



Our final problem, Example 7.2.3 below, usually takes students an entire two-hour seminar to complete a thorough investigation of. This example is not so much a 'question', but an open-ended investigation which allows students to develop not only their problem-solving skills, but their ability to specialise and generalise mathematical ideas.

▼ Example 7.2.3 — *Fibonacci Factors*

The Fibonacci sequence F_n is defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all positive integers n.

Look at what factors the sequence have. By choosing different starting numbers, or the recurrence relation we can generalise Fibonacci sequences, e.g. Lucas numbers have $L_1 = 2, L_2 = 1$ and $L_{n+2} = L_{n+1} + L_n$; Pell numbers have $P_1 = 1, P_2 = 2$ and $P_{n+2} = 2P_{n+1} + P_n$.

Outcomes and the Future

Having no previous experience of university level mathematics, students in general take very well to the module. There are always a few who fail to grasp the ideas contained in *Problem*

Solving, for whom the module is less enjoyable, however these students make up a small minority of the cohort.

Having led the module for the last five years, Adrian Simpson is moving to the teaching of a different module next year, and so its leadership will be passed onto another member of staff, albeit one who has taught *Problem Solving* before.

7.3 University of Leicester

In 2009/10 the University of Leicester's second-year mathematics module *Investigations in Mathematics* was taken over by Dr. Alex Clark and Prof. John Hunton who, with colleagues in the Department of Mathematics, decided to teach problem-solving using a modified Moore Method. A range of topics from both pure and applied mathematics are offered to students, who work in groups to solve a sequence of problems.

Module	Investigations in Mathematics
Year	2nd
Code	MA2510
Hours	1 hour a week
Length	One term (10 weeks)
Mandate	Compulsory for single-honours students
First run	2009/10[1]

Teaching

There are around 100 second-year, single-honours mathematics students at Leicester, all of whom take *Investigations*; it is not an option for dual-honours students. Five topics are offered on the course, ranging from graph theory to communication theory, and students choose which they sign-up to on a first-come, first-served basis. There are nine groups in total, each topic but one is covered by two groups; each group has between six and twelve students in it.

The teaching method is derived from the Moore Method and proceeds in a similar manner to that of Birmingham's Moore module (see Section 7.1), with the notable addition of group work. Students work on problems outside classes, and during classes present solutions to problems on the board. As always, it is the intention that the teacher, besides choosing who presents at the board, says as little as possible during class, and when doing so only ask questions. Some members of staff found themselves deviating slightly from this plan of action, offering students more direction at times, though the emphasis is still strongly on problem-solving and *students* communicating mathematics for themselves.

Having a single, one-hour class each week, the expectation is that students come to class with solutions to problems prepared. In class, a student will go to the board to present a solution to a problem, and others will give assistance or make suggestions if the student is stuck or there are improvements to be made. Such presentations are not expected to be polished, as one would find in a standard mathematics lecture, but a part of the process for creating a final, written, submission of correct mathematics. It is the teacher's choice who presents at the board; if a student repeatedly declines to present a solution, they may be asked to be a 'scribe' – writing a solution on the board that is based on the work of the whole group.

[1] Previously run under the same name and code, but not in its current form.

Students are not prevented from looking in textbooks or online for help with problems, however there are a number of approaches taken to make doing so less helpful than it may otherwise be. Any names attached to theorems are revealed to students only after they have completed the module, and problems are frequently so specific as to make searching online for solutions a fruitless activity. While teaching chaos theory, Dr. Clark used an unconventional definition of fractal dimension that made it easier to spot had students lifted work from elsewhere.

Written submissions to problems take the form of notebooks, which are assessed mid-term and at the end of the semester. When it first began, students were not required to complete any assessment until the end of the module, and there were a number of instances of students having no work to present during class. A poster (submitted after three weeks) and an interim notebook (submitted after six weeks) were introduced after the first year of the module to encourage early engagement with the material, and have been successful in improving participation.

Group Work

Students are encouraged to work together in solving the problems in *Investigations*, and further collaboration takes place when students present problems in classes; however group work is not strictly an objective of the module, and students may work alone if they wish. There is the possibility that some students will take advantage of the group environment and rely on others to solve problems for them, though the majority of assessed work is still the notebooks in which they write up the solutions to problems. Frequently the groups that the students work in are subsets of their subject group; i.e. they work in twos and threes as opposed to ten together.

The poster in the third week and a presentation, given at the end of term, are the two places where subject groups have to work together. More of this is made in the following section.

Assessment

Unlike the majority of courses in the department, there is no final exam, rather assessment is entirely dependent on work done inside and outside of classes. The make-up of a student's mark for the module is given in Table 7.1.

The contribution and participation mark, worth 20% over the course of the module, is awarded to students for engaging in the discussion of ideas and solutions in classes. It exists to encourage both this discussion, and work outside of class.

Assessment	Assessed	Percentage
Meeting contributions	All term	20%
Poster	Week 3	10%
Interim notebook	Week 6	10%
Presentation	Week 10	20%
Final notebook	Week 10	40%

Table 7.1: Distribution of marks in *Investigations in Mathematics*

The poster and interim notebook also encourage students to work continuously throughout the module, and allow staff to give feedback before the submission of the final notebook. Notebooks consist of the students' answers to the problems in the module, interwoven with further explanatory material they have researched to embellish the topic they have chosen. The expectation is that the mathematics in the notebooks is of a high standard.

Each subject group collaborates on the poster and presentation; for presentations the nine subject groups are divided into three sets of three groups. Each group then gives a presentation to the two other groups in its set; sets are chosen so that no two groups in the same set cover the same topic. Presentations last 15 minutes each, with a further 5 minutes for questions; it is not expected that each student will talk in the presentation itself, though the work to write the presentation has to be a collaborative effort. Presentations are marked out of 20; half of the marks for the degree to which they accurately cover the important aspects of the material the students have studied, and the other half for presenting the material in an engaging and understandable way.

Motivations and Development

The module has a number of aims, two of which particularly stand out. The first is that students have the opportunity to see mathematics developed from the ground up, and in doing so are engaged on a more fundamental level than one would find in a standard lecture-based module. The second objective is that the module develops students' skills in communicating mathematics rigorously, be it verbally or in writing. Teaching a part of a topic in mathematics, while not the central point of the module, is certainly an aim; for that reason the topics that are taught are important. It is difficult to teach a certain topic in this manner if students have serious gaps in the basic knowledge that they require to begin studying it.

Investigations in Mathematics existed for several years before 2009/10, and had gone through a number of iterations in its approach to teaching. While problems were previously used in the module, they did not fit to an overarching theme, nor lead students through a topic in any well-defined manner. It is this approach to structured problem-solving which defines the current version of *Investigations* as different from its previous incarnations. This approach was precipitated on the experience of Dr. Clark as a student in the United States, who had been taught using the Moore Method as an undergraduate. The intention was not to emulate the Method closely but to develop a teaching process based on Dr. Clark and Prof. Hunton's combined experience.

In the second year of the new approach the presentation and midterm notebook were introduced to encourage students to work on the module continuously throughout the term. While only accounting for 10% of the module's mark each, this change lead to an improvement in student participation during classes, and their marks for the final notebook submission.

Problem Selection

Problems for the module are usually written by the individual members of teaching staff, though if a single topic is taught by two people, one will use the notes of the other. The following excerpt is from Dr. Clark's group on fractals:

▼ Example 7.3.1 — *Fractals*

In order to analyse fractals on a small scale, we need a way to zoom in on them and to measure them on the small scale. That is the motivation behind the following notions. We eventually want to analyse fractals in $\mathbb{R}$, $\mathbb{R}^2$, and more generally in $\mathbb{R}^n$. Each of these requires its own tailor made notion of zooming.

Definition 1.6 *For $\mathbb{R}$ and $\delta > 0$, the δ-mesh is the collection of intervals*

$$\{[n \cdot \delta, (n+1) \cdot \delta] : n \in \mathbb{Z}\}.$$

Definition 1.7 *For $\mathbb{R}^2$ and $\delta > 0$, the δ-mesh is the collection of squares*

$$\{[n \cdot \delta, (n+1) \cdot \delta] \times [m \cdot \delta, (m+1) \cdot \delta] : m,\ n \in \mathbb{Z}\}.$$

Problem 1.8 *For $\mathbb{R}^3$ and $\delta > 0$, define the δ-mesh. Likewise for $\mathbb{R}^n$.*

This is a typical example of the written content of the module: definitions and problems, with few examples. In other cases, progressions of problem have been taken from the *Journal of Inquiry-Based Learning in Mathematics*, which has progressions of problems in a range of mathematical topics, designed specifically for teaching Moore Method modules.

▼ Example 7.3.2 — *Coding theory*

Using the language with just two letters (say 0 and 1), find examples of codes which (a) detect 1 error, and (b) detect and correct 1 error, when you have three *distinct objects. What about 4 (or more) objects? Can you see any general results? What about more than one error? Can you find shorter codewords when you have 3 (or 4, or more) letters in your language?*

Examples 7.3.2 and 7.3.3 are taken from Prof. Hunton's coding theory group, and show the progression of problems in the module from basic introduction to more involved mathematics.

▼ Example 7.3.3 — *Finite fields warm-up*

1. *Recall the definition of a* field *from MA2103.*
2. *Why is $\mathbb{Z}/m$ a field if $m > 1$ is a prime, but not if it is not a prime?*
3. *Suppose $z(x)$ is a polynomial in x over a field $\mathbb{F}$ of degree r, that is, it is an expression*

   $$a_0 + a_1 x + \cdots + a_r x^r,$$

   for some $a_i \in \mathbb{F}$. Say $z(x)$ is irreducible over $\mathbb{F}$ *if there you cannot write $z(x) = q(x)s(x)$ for polynomials over $\mathbb{F}$ $q(x)$ and $s(x)$, both of degree at least 1. What are the irreducible polynomials of degree 2 over $\mathbb{Z}/2$?*
4. *For what (small) primes p is $x^2 + 1$ irreducible over $\mathbb{Z}/p$?*
5. *The construction of a field $\mathbb{F}_{p^r}$ with p^r elements is as follows. Let $z(x)$ be an irreducible polynomial of degree r over $\mathbb{Z}/p$, and take $\mathbb{F}_{p^r}$ to be the set of all polynomials in x over $\mathbb{Z}/p$ subject to the rule $z(x) = 0$: think of this as the analogue of constructing $\mathbb{C}$ by taking $\mathbb{R}$ and a new 'number' x (normally called i) subject to the rule $x^2 + 1 = 0$. Construct the addition and multiplication tables for the field with 4 elements. What about the fields with 9 or 25 elements?*

Outcomes and the Future

As with any problem-solving module that relies largely on students to solve problems for themselves, students progress through material very slowly when compared to standard lecture-based modules. It is not the aim of the module to teach a complete area of mathematics, however, and it is felt that students do digest the material better in *Investigations* than in other modules. Students are generally enthusiastic about the module and the feedback is more positive than most modules, though a few students struggle with the approach, especially in the event that their grasp of English is limited.

It has been decided that, from next year, the module will be taught by two dedicated members of staff, instead of the six or more that were previously used. This is preferable from an administrative point of view – co-ordinating the teaching commitments of six members of staff, some of whom teach other modules besides *Investigations*, had proven problematic. A further advantage is that those staff who will be teaching the module are more experienced with it, rather than having to use simply the members of staff that are available in that term, whatever their thoughts on the approach.

7.4 University of Manchester

Module	Mathematics Workshop
Year	1st
Code	MATH10001
Hours	3 hours a week
Length	One term (10 weeks)
Mandate	Compulsory
First run	2004/05
Uptake	270 students

The first-year, first semester, module *Mathematics Workshop* has been running at the University of Manchester since the merger of Victoria University of Manchester[2] and University of Manchester Institute of Science and Technology (UMIST) in 2004. It aims to develop students' modelling and problem-solving skills, to improve their ability to write and present mathematics, and to teach them to use the software MATLAB. *Mathematics Workshop* is compulsory for all 270 single and some joint honours mathematics students, who take the module in ten separate groups, each led by a postgraduate facilitator. Run by Dr. Louise Walker, the module is described in the first-year handbook as follows:

> "These weekly classes are intended to help students experience a wide range of mathematical topics. The course unit includes a number of projects to be worked on individually and in groups. The projects are assessed via a written report. Marks will be awarded for presentation as well as mathematical content to encourage the development of good writing habits.

Teaching

Mathematics Workshop consists of a one hour lecture and a two hour workshop every week; the style in which the workshops are taught varies over the course of the module. Its content is as follows:

- Week 1: Introduction to the course
- Weeks 2–5: Introduction to MATLAB and numerical methods
- Week 6: Mid semester break
- Week 7: Introduction to modelling and problem-solving
- Weeks 8–9: Project 1
- Weeks 10–11: Project 2
- Week 12: Test

During the first five weeks of the module, workshops are held in computer labs where students work on individual problems. Here, the focus is not so much on problem-solving but

[2] Victoria University of Manchester was colloquially known as the University of Manchester.

rather becoming familiar with using MATLAB, which students then use later in the module to help solve the problems they are given. During these weeks, assessment is by worksheets in workshops and homeworks, each of which are marked by hand by the postgraduate facilitators. The following is an example of the MATLAB homework in *Mathematical Workshop*, taken from the third week of the module:

▼ Example 7.4.1 — *MATLAB Homework*

Below is the code used to generate the picture of the Mandelbrot set given in the lecture. Use MATLAB's help command to try and understand what each of the lines of code is doing. You may also need to look up the definition of the Mandelbrot set.

```
x=-2:0.01:0.6;
y=-1.1:0.01:1.1;
[X,Y]=meshgrid(x,y);
C=complex(X,Y);

Z=C;
for k=1:50
    Z=Z.^2+C;
end

contourf(x,y,abs(Z)<10^5)
```

Following the mid-semester break, the focus of the module shifts to modelling and problem-solving. After an introductory week, students complete two group projects lasting two weeks each, covering topics from applied mathematics. In 2011/12, the first project covered difference equations; the second, graph fitting. Both *How to Solve It* (Pólya, 1945) and *Thinking Mathematically* (Mason et al., 2010) are set texts for the module, and students are referred to relevant chapters in the latter book at appropriate times in lectures, to aid their understanding of the process of solving unfamiliar problems.

Lectures cover mathematical material, and include a basic example of the type of problems included in workshops. Workshops include two problems, the first of which will be a relatively straightforward example close to those seen in lectures; the second of which relies on modelling and an exploration of the model's limitations. Example 7.4.2 gives the two problems from the first week of the project on difference equations.

▼ Example 7.4.2 — *Difference Equations*

1. *Consider the second order difference equation*

$$y_{n+2} - 2y_{n+1} - 3y_n = 0,$$

with $y_0 = 3$ *and* $y_1 = 5$.

   (a) *Find* y_2 *and* y_3.

   (b) *Find the solution of the difference equation and check that your formula gives the correct values for the first 4 terms of the sequence.*

   (c) *Write down an expression for* $y_{n+1} = y_n$. *Show that as* $n \to \infty$, $y_{n+1} = y_n \to 3$.

2. *We want to study the populations of rabbits and foxes that live in a certain region, from a given starting point. Let the population of rabbits after n months be denoted by* r_n *and the population of foxes be* f_n. *The initial populations are* r_0 *and* f_0.

   We model the populations using the following coupled first order difference equations:

$$r_{n+1} = ar_n - bf_n$$
$$f_{n+1} = cf_n + dr_n$$

where $a; b; c; d > 0$ *are parameters.*

   (a) *Explain why these equations seem a reasonable way to model the problem. What real life properties of the problem could the parameters* $a; b; c; d$ *represent?*

   (b) *Eliminate* f *from the set of equations to find a single linear second-order difference equation in* r. *Solve this equation to get a formula for* r_n *in the case when* $a = 2$; $b = 1 : 3$; $c = 0 : 5$ *and* $d = 0 : 4$ *with initial populations* $r_0 = 500$; $f_0 = 100$.

   (c) *What are the limitations of this model?*

These problems are answered by students during the two hour workshop, working in groups. Most groups manage to complete draft solutions in the time available, though frequently take half an hour or more to get started. Postgraduate facilitators spend time with each of the five or six groups in their room to ensure that students understand the problems and are getting on with work.

For homework, students complete the solutions to the problems given in workshops, which form part of their project report (see the following section on assessment for further details), and also complete a separate homework task. Frequently, this homework task involves using MATLAB to model scenarios from the second worksheet problem, which is then submitted as a .m file for assessment. Finally, completing the project report is included as homework for the two fortnightly projects.

Group Work

Group work, and learning to work in groups, is central to *Mathematics Workshop*. While students submit individual reports for each of the projects that they complete during the module, 5 of the 30 marks for each project are from the average for the group. Students are warned that:

> “ Even though you have worked as a group on this project, the report should be **all your own work**. There are marks for the clarity as well as the correctness of your mathematical arguments.

Assessment

The majority (60%) of the marks for the module are derived from the project reports that students submit after each of the four fortnightly projects; these are marked by the postgraduate facilitators and reviewed by Dr. Walker. The remainder of the marks are awarded for MATLAB assignments and the class test in the last week of term. The breakdown of marks is as follows:

- MATLAB Assignments – 30% total
- Two project reports – 30% each, comprising:
  - 20% for solutions to problems
  - 3% for homework
  - 2% for the presentation of the report
  - 5% from the group mean of the first three items
- Class test – 10%

The majority of the content from the project reports is solutions to the worksheet problems.

Motivations and Development

Mathematics Workshop is derived from a previous module that was taught to first-year engineering students at Victoria University of Manchester. In 2000, the School of Engineering began teaching its first-year undergraduates primarily using problem-based learning, and the mathematics service module provided by the School of Mathematics was required to conform to the new standard. There the format of one one-hour lecture, one two-hour workshop was begun, and proved popular with students who had previously learned mathematics from a standard lecture-based module.

Dr. Walker had been involved with the service module from its inception, and when Victoria University of Manchester merged with UMIST in 2004, she was keen to introduce a module along similar lines for first-year mathematics students. As a result of the two universities' merger, the mathematics degree programme was completely redesigned and finding space within an existing programme was not an issue. When the teaching committee of the new department came to structure the first-year of the various mathematics degrees, it was recognised that previous provision for problem-based learning and the development of skills outside core technical competencies was lacking.

The aim of the module is not to teach a particular area of mathematics, but to give students an apprenticeship in doing mathematics for themselves. For many, the use of MATLAB in the module will be their first experience of programming, and so the workshops introduce a range of new ideas and approaches to students. Furthermore, both the group-based teaching and project-based assessment of the workshop satisfy a number of internal assessment criteria for the undergraduate mathematics degree programme as a whole, specifically the departmental employability audit.

Problem Selection

Both the MATLAB and group problems are written specifically for the module; the topics for which are chosen each year by Dr. Walker. Problems in early iterations of the module frequently proved too challenging for students, who required more help than expected and were often frightened into inaction. Since then the difficulty of problems has been reduced, in order to encourage students to complete the work as much as possible without intervention from teaching staff; weaker students may still rely on their stronger peers in a group, however. The module is now less ambitious than when it began; students find tackling problem-solving enough of a challenge even with relatively straightforward mathematics.

This year the module had no pure mathematics problems, partly as a result of its involvement with a HE STEM Programme project on modelling, run by Prof. Mike Savage at the University of Leeds; next year a pure topic will be reintroduced to broaden the range of mathematics that students get to study in the workshops.

Outcomes and the Future

Students, in general, enjoy the module and appreciate what it aims to achieve; as always, some are uncomfortable with the module's approach and prefer traditional lecture-based teaching. Each year, every module in the department is assessed by a focus group to address concerns and improve outcomes for future students. While a self-selecting group, of this year's students six were general positive, one neutral and one negative about the module. Negative comments about the workshops frequently refer to the open-endedness of the problems and their general unfamiliarity.

Managing the module takes more effort on the part of its leader than a standard lecture module, but it is felt that this investment of time is very worthwhile, given its outcomes. Besides the reintroduction of pure mathematics in its next iteration, it is anticipated that the module will continue in its existing form for the foreseeable future.

7.5 Queen Mary, University of London

Module	Mathematical Problem-Solving
Year	3rd
Code	MTH6124
Hours	1 hour a week
Length	One term (11 weeks)
Mandate	Optional
First run	2004/05

Mathematical Problem-Solving is a third-year undergraduate module at Queen Mary, University of London. It aims to develop students' problem-solving skills, their creativity and logical thought processes, and proof writing. While the module does not have explicit prerequisites, certain of the problems that students solve require knowledge from core second-year modules. *Mathematical Problem-Solving* is an optional module, taken by around a dozen students a year and currently taught by Dr. David Ellis. The module is advertised to students in the third-year handbook as follows:

> "The module is concerned with solving problems rather than building up the theory of a particular area of mathematics. The problems are wide ranging with some emphasis on problems in pure mathematics and on problems that do not require knowledge of other undergraduate modules for their solution. You will be given a selection of problems to work on and will be expected to use your own initiative and the library. However, hints are provided by staff in the timetabled sessions. Assessment is based on the solutions handed in, together with an oral examination.

Teaching

Essentially a second semester reading module, each student who takes *Mathematical Problem-Solving* works through a set of a dozen problems taken from a range of topics in pure mathematics. To prevent plagiarism, each pupil is given a different set of problems, and problem sets have roughly equal numbers of problems on geometry, combinatorics, analysis and number theory. Complexity theory and combinatorial geometry problems are also present in some problem sets. Before beginning the module, students are given a few practice problems to work on over the Christmas holiday, to give them an idea of the structure and difficulty of the module.

Example 7.5.1 is a number theory question from the module that is worth 8 marks, the sixth in its particular progression.

▼ Example 7.5.1

Do there exist three natural numbers $a, b, c > 1$, such that

$a^2 - 1$ *is divisible by b and c,*

$b^2 - 1$ *is divisible by a and c, and*

$c^2 - 1$ *is divisible by a and b?*

A one-hour class takes place each week, where students can get direction from the module leader on particular problems, or work on problems while others ask questions. Students are encouraged to contact the module leader in person and by email if they require assistance outside of class, which many choose to do. Other members of staff may offer students assistance with a problem, in which case they report to the course leader exactly what assistance was given.

Students receive problems individually, with five or six days between problems; if they submit a solution to a problem early, they will be given the next problem to work on. Solutions can be submitted to the module leader at any time, and these will be returned with appropriate comments. If there is an issue with a student's solution they can work further on it and resubmit their solution with no mark penalty.

At the beginning of the module it is explained to students that the problems they will be presented with during *Mathematical Problem-Solving* are not expected to be solved within a few minutes. This is frequently at odds with their previous mathematical experience of working on an individual question for perhaps no more than an hour, and is designed to give students a flavour of what it means to do mathematical research. Related to this, the solutions which students submit are expected to be mathematically complete, to a reasonable degree as defined by the module leader.

Group Work

There is no group work in the module.

Assessment

Each problem in *Mathematical Problem-Solving* is worth between 6 and 17 marks, a student is awarded those marks if they produce a correct solution. Marks are given for partial solutions, and the marks for a correct solution may be reduced in cases where students are given significant assistance. The final mark for the module is the sum of the marks gained from a student's ten highest scoring solutions, thus if a student's ten highest scoring answers are worth seven marks each, they get a score of seventy. The distribution of available marks means that to get 100% for the module a student will need to answer several questions of considerable difficulty worth 13 or more.

Final solutions to all twelve problems are submitted at the end of the first week after Easter. Thus, students will have twelve weeks working on the problems during term time, and four further weeks of the Easter holiday to refine their solutions (though no assistance is given by staff once the Easter break begins).

After submitting their solutions, and at the beginning of the exam period after Easter, students are given an individual oral examination to ensure that they understand their solutions and that their work is their own. No marks are awarded for the exam, though it gives the opportunity for the module leader to clarify ambiguities in a students' solutions and adjust marks accordingly, if needed. The oral examination normally takes a quarter of an hour; there is no written examination.

Model solutions to problems are never handed out, the only solution a student sees is the one they produce.

Motivations and Development

Mathematical Problem-Solving has been running in essentially its current form for over 20 years, though the module code and some of the problems it uses are different from the originals. Professor Charles Leedham-Green created the module in the late 1980s to give students an experience of learning mathematics by a means other than lecture-based modules; the purpose and format of the new module was agreed by the then Head of Department as teaching committees did not exist at the time.

When the module began, each student completed the same set of problems; after a time it was realised that the most effective way to combat potential plagiarism was to give each student a different set of problems. To achieve this, Prof. Leedham-Green and Dr. Susan McKay spent a considerable amount of time scouring books and British Mathematical Olympiad papers to find problems that were suitable; there is no record of where individual problems were taken from.

During its early years, the module was the only one in the department that consistently required no scaling; furthermore students frequently produced solutions to problems that were unique, a rare occurrence in other modules.

In the current iteration of the module from January 2012, there was initially no set class time and students were encouraged to visit staff individually. However, during the first weeks of the module many students expressed a desire for a set class time, and so a class hour on Thursday afternoons was introduced in February.

Problem Selection

The problems currently used in *Mathematical Problem-Solving* are a superset of those found by Prof. Leedham-Green and Dr. McKay; added to over the intervening years by the members of staff who have lead the module. Individual problems generally consist of one or two parts; in problems with two or more parts, these individual parts are related in some way. The following is an eight mark question with two parts; the first of which is relatively straightforward but which prompts one to consider possible solutions to the second part:

▼ Example 7.5.2

Find all positive integers n such that $(n-1)!$ is not divisible by n. Repeat this for n such that $(n-1)!$ is not divisible by n^2.

The following example is worth 15 marks, and presents a considerably greater challenge to students. Here it may not be immediately clear what the question is asking:

▼ Example 7.5.3

If $2n$ points are evenly placed on the circumference of a circle, and then a (non-convex) $2n$-gon is drawn joining up these points, prove that it must have two parallel sides.

Outcomes and the Future

The module is aimed at the strongest students, and yet there is a range in success in solving problems. Most students who take the module perform relatively well and a few excel, requesting new problems frequently and scoring close to full marks. Some struggle with the module, however, while others are put off by the practice problems over Christmas. Students who need the most assistance accept that they will not get full marks for a correct solution to a problem, and so their potential mark for the module as a whole will be lower.

The six students who have completed the current year's module assessment form either mostly or strongly agreed that the module was well taught, with clear marking criteria, and adequate feedback and support being given by lecturers. This feedback mirrors the response of previous years' students. The module is popular among teaching staff, not least because, with a good bank of problems from which to work, the time required to teach the module is much less than that of a standard lecture course. Furthermore, staff recognise the benefits that the module brings to student who take it and work hard.

7.6 University of Warwick

Module	Analysis 1
Year	1st
Code	MA131
Hours	5 hours a week
Length	One term (10 weeks)
Mandate	Compulsory
First run	1996/97

Analysis 1 is a first-year, first-term, analysis module at the University of Warwick, taken by all 320 single-honours mathematics undergraduate students and taught using a problem-solving approach. The module is followed in the second-term by *Analysis 2*, a traditional lecture-based module.

Analysis 1 was the subject of Alcock and Simpson's chapter *The Warwick Analysis Project: Practice and Theory* (Alcock and Simpson, 2001), that explored the origins of the module, the experiences of students and teachers, and its efficacy. Today the module is somewhat different from its original form, though the general approach of teaching by problem-solving is very much alive.

The first-year handbook advertises the module to students as follows:

> “With the support of your fellow students, lecturers and other helpers, you will be encouraged to move on from the situation where the teacher shows you how to solve each kind of problem, to the point where you can develop your own methods for solving problems. You will also be expected to question the concepts underlying your solutions, and understand why a particular method is meaningful and another not so. In other words, your mathematical focus should shift from problem solving methods to concepts and clarity of thought.

Teaching

The module is taught with a single, one-hour lecture and two, two-hour classes every week. Lectures proceed in the traditional manner, though given that there is only one lecture a week, time during them is spent on the most important concepts and difficult proofs that students will have to tackle during classes. Each week students are given a workbook of roughly ten pages, and a set of assignments to complete for handing in the following week.

A single lecture caters for all 320 students who take the module; in classes students are divided arbitrarily at the beginning of the module into groups of around 30. Each group is assisted by a postgraduate student or member of teaching staff, and a second- or third-year undergraduate student, whose jobs are to prompt students that are stuck, to answer questions (without revealing answers), and to encourage students as necessary. The first class of a week is before the lecture, and so students first experience the module material in classes.

Workbooks are distinguished from traditional lecture notes by much of the theory being contained in the assignments that students complete. Example 7.6.1 is taken from the second page of the fifth workbook, covering completeness:

▼ Example 7.6.1 — *Completeness*

Theorem

Between any two distinct real numbers there is a rational number.

[i.e. if $a < b$, there is a rational $\frac{p}{q}$ with $a < \frac{p}{q} < b$.]

Assignment 1

Prove the theorem, structuring your proof as follows:

1. *Fix numbers $a < b$. If you mark down the set of rational points $\frac{j}{2^n}$ for all integers j, show that the one lying immediately to the left of (or equal to) a is $\frac{[a2^n]}{2^n}$ and the one lying immediately to the right is $\frac{[a2^n]+1}{2^n}$.*
2. *Now take n large enough (how large?) and conclude that $a < \frac{[a2^n]+1}{2^n} < b$.*

Corollary

Let $a < b$. There is an infinite number of rational numbers in the open interval (a, b).

Assignment 2

Prove the corollary.

We have shown that the rational numbers are spread densely over the real line. What about the irrational numbers?

Group Work

While the entire cohort is initially divided into groups of 30, further subdivision happens between students organically. Students are encouraged to work in smaller groups however, and over time this generally results in three or four students working together in classes. Each group is visited either by the postgraduate or undergraduate assistant to make sure that they are on task.

Assessment

For the purposes of assessment, *Analysis 1* and *Analysis 2* are considered a single, two-term module. The majority of the marks for this combined module, 60%, are awarded for the three-hour end-of-year examination, which includes some questions relating to *Analysis 1*, though mostly focused on the second-term. The total credit for the module is made up as follows:

- *Analysis 1* – Mid-term exam: 7.5%
- *Analysis 1* – Final Examination: 25%
- *Analysis 2* – Weekly assignments: 7.5%
- Final examination: 60%

The *Analysis 1* final examination is taken in the first week of the second-term and lasts 90 minutes; the hope is that most students score 60% or more, though many fail to reach that mark. Time frequently factors in the examination—students overestimate what they can get done—and, being their first formal exam of the year, it serves as a wake-up call for many. The only other examination students take at the same time is for *Foundations*, a module that covers a range of mathematics from prime factorisation to induction, which is considered more straightforward than *Analysis 1* by most students. The mid-term exam is a less formal affair; taken by students during a two-hour class, the test covers the first five weeks' content.

Besides the mid-term and final examinations, students have weekly assignments in *Analysis 1* that they hand in at the start of class on the following Monday. These assignments are just those from the workbooks themselves, numbering around sixteen (though there a only eight in the first week).

Motivations and Development

The origins of *Analysis 1*'s existence as a problem-based learning module are documented by Alcock and Simpson (2001):

> “ Recognising that students continued to perform poorly despite his efforts at improved presentation, one member of the Warwick staff with over 30 years of traditional teaching experience chose to adopt a new lecturing style. In an initial modification of a second-year metric spaces course, he encouraged students to develop much of the mathematics for themselves by posing central, illuminating questions. While this course met with limited success and even more limited appreciation from the students, he persevered to initiate the Warwick Analysis Project, approaching two members of the mathematics education group to help teach it in the first-year.

> The project is based on students' completion of a carefully structured sequence of questions, through which the syllabus of the Analysis 1 course is developed. An excellent textbook bringing together mathematical, pedagogical and mathematics education research knowledge had been developed over a number of years at another university and this text (Burn, 1992) is central to the course.

In 1996/97 the new-style module was piloted with 35 volunteer students and, after positive results, was taken by all 235 single-honours mathematics students from 1997/98. Joint-honours students, numbering around 150, continued to be taught in the traditional manner, and as a result comparisons could be drawn between the examination performances of the two cohorts.

	Foundations	Analysis 1	Analysis 2
New Module Students	60.0	76.5	63.5
Traditional Module Students	57.7	64.6	51.1

Table 7.2: Examination results in Alcock and Simpson (2001, p. 106)

Examination results for single- and joint-honours students taking the new- and traditional-style modules are given in Table 7.2. The *Foundations* and *Analysis 1* examinations were those taken in the first week of the second-term (the same exam structure remains today); the *Analysis 2* examination is from the third term. Comparisons between assessments within these modules were not made because 1) assignments were different in the new and traditional modules, and 2) both analysis modules are counted as one for the purposes of a final mark. While one may expect the examination marks of single-honours students to be higher than those of joint-honours students, the difference is dramatic in the analysis modules, and furthermore *Foundations* marks—a module where both cohorts are taught in the same manner—are much closer to one-another.

Problem Selection

As mentioned in the previous section; the first few iterations of the new-style *Analysis 1* used Burn's *Numbers and Functions: Steps into Analysis* (1992) as the source of the workbooks and1 assignments. Original notes were then written for the module by Alyson Stibbard, who was at the time a member of the teaching staff at Warwick; these workbooks better streamlined the student experience for the realities of time and the curriculum at the

Mathematics Department. These workbooks have been used, largely unchanged, since 1998/99.

The problems themselves come directly from the material being taught; they are not selected so much as they are necessary elements of the process. The topics covered by the module are:

- Inequalities – Manipulation; power and absolute values; the Bernoulli and triangle inequalities.

- Sequences – monotonic and bounded sequences; subsequences; limits and convergence; sandwich theorems; recursively-defined sequences.

- Real Numbers – numbers in an open interval; rationals/irrationals and decimals; sets and bounds; completeness and its consequences; k^{th} roots; Bolzano-Weierstrass theorem; Cauchy sequences.

- Series – partial sums; convergence and divergence; sum and shift rules; boundedness condition; comparison, ratio, integral, and alternating series tests; absolute and conditional convergence; rearrangements.

Besides the sections of exposition and problems that build up students' understanding of analysis, all but the first workbook finish with one or two sections on applications. For instance, the fourth workbook covers more advanced sequences (continuing on from Workbooks 2 and 3), and ends with a section on the application of the theory of sequences of numbers to sequences of more general objects, in the case of Example 7.6.2, polygons.

▼ Example 7.6.2

Example *Let P_n be the regular n-sided polygon centred at the origin which fits exactly inside the unit circle. This sequence starts with P_3, which is the equilateral triangle.*

Let $a_n = a(P_n)$ be the sequence of areas of the polygons. In Workbook 1 you showed that $a_n = \frac{n}{2}\sin(\frac{2\pi}{n})$ and in Workbook 2 you showed that this converges to π as $n \to \infty$ which is the area of the unit circle.

Exercise 6 *Let $p_n = p(P_n)$ be the perimeter of P_n. Show that $p_n = 2n\sin(\frac{\pi}{n}) = 2n\sin(\frac{2\pi}{2n}) = 2a_{2n}$.*

Assignment 14 *Show that $\lim p_n = 2\pi$.*

Outcomes and the Future

Analysis 1 is currently taught by Dr. John Andersson, assisted by a team of undergraduates and postgraduates from the Mathematics Department. Dr. Andersson has taught the module for the last two years, and is keen to do so again; this, in spite of the initial feelings of discomfort with a style of teaching so different to the norm. The level of interaction with students is much higher than that of a normal lecture module, and though students may at first be reticent to engage, this improves over the term that the module is taught. As Dr. Andersson says:

> “ I think in general that it's impossible to teach people mathematics; I think the only way you learn mathematics is by doing it. However, I think enthusiasm is important, because they don't have to turn up at lectures. If your lectures aren't interesting then they won't turn up, so hopefully you have something to say in the lectures that can help them in the right direction, in developing.

While the module's source material has changed since slightly its inception (though not for several years), the teaching method is largely unmodified, and, crucially, single-honours mathematics students still perform appreciably better (an estimated 7-8%) than joint-honours students.

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Appendix A

GeoGebra Worked Examples

Santiago Borio Peñaloza

The following examples, created for this project by mathematics teacher Santiago Borio Peñaloza, show some of the capabilities of the free and open-source software GeoGebra (www.geogebra.org). Originally a geometry package, GeoGebra is a powerful tool for investigating phenomena in a range of topics in mathematics; suitable for use by both students and teachers. The files for the examples shown here, and three further example problems, are available on the project website at www.mathcentre.ac.uk/problemsolving. Three further examples available are:

Rotations in Polar Coordinates An introduction to the concept of polar coordinates and an exploration of some of their uses and applications.

Isosceles Spirals Considering aspects of spirals generated by isosceles triangles, leading to the Spiral of Theodorus and the Golden Spiral.

An Application of the Intermediate Value Theorem Show that any planar polygon can be bisected by a single straight line. This is a well know problem that appears in various forms and contexts.

A.1 Problem 1 – Constructions

Statement of the Problem

Given a triangle, construct a square with its vertices lying on the edges of the triangle. Is it always possible to construct such a square? If so, explain how.

Investigation and Solution

By the pigeon-hole principle, two vertices of the square must lie on one side of the triangle and then the other two vertices must lie on the two remaining sides. Furthermore, if the triangle has an obtuse angle, the side containing two vertices of of the square must be the side opposite the obtuse angle.

Label the vertices of the triangle ABC with angle ACB the largest. (The inscribed square we are trying to find will eventually be labelled $DEFG$.) Choose a point D on the side AB. By drawing in turn three suitably orthogonal lines DF, FG, and GE as shown in Figure A.1.1, we obtain an inscribed rectangle $DEGF$. When D is close to A, the side DE is the longest side of the rectangle. As D moves towards B, the ratio of the sides changes until a point is reached where DF is the longest side. A geometric version of the *Intermediate Value Theorem* shows that during this transition the rectangle will pass through a square stage. GeoGebra makes this visually convincing by allowing the user to see the rectangle transform as the point D is dragged back and forth along AB.

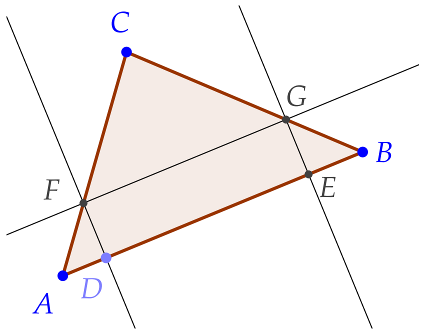

Figure A.1.1:

Having seen, by general considerations, that an inscribed square must exist, we now need to find a way of constructing it. Consider Figure A.1.2, which shows the triangle ABC with the square $DEFG$ inscribed; in particular, observe that ABC is similar to the triangle FGC.

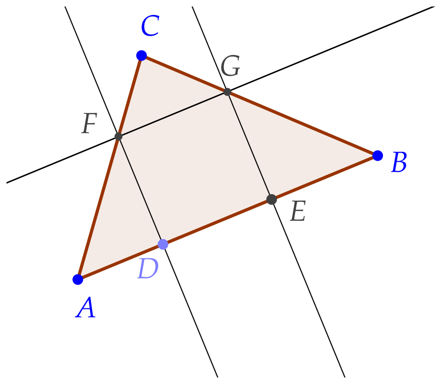

Figure A.1.2:

Now focus on FGC with the square $DEFG$ attached to its base. Using a radial change-of-scale transformation centred on C, we can blow up the triangle FGC to ABC so that the square on its base transforms to a square $PQAB$ on the base of ABC as shown in Figure A.1.3.

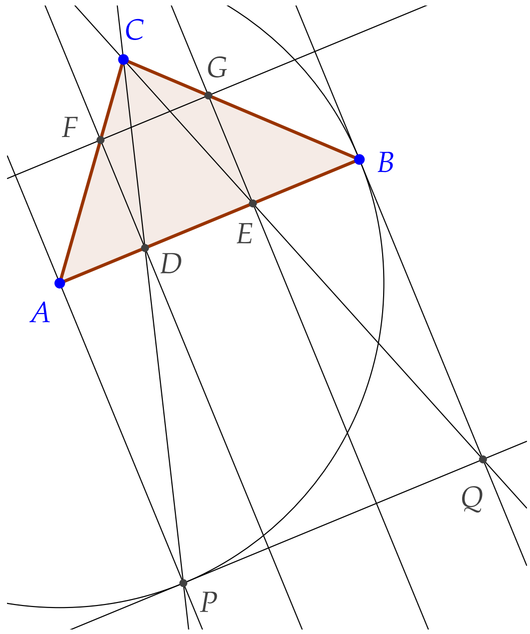

Figure A.1.3:

Therefore, in order to find the points D and E that will define our inscribed square $DEFG$, we first construct the square $PQAB$ on the base of triangle ABC, and then construct D and E as the intersections of CP and CQ, respectively, with AB. Once again, the constructions can be animated in GeoGebra by dragging various points in the figure.

Interesting Points and Similar Problems

This problem, when considered formally, requires careful justification using ideas which may not be familiar to students (pigeonhole principle, similarity, etc). However, the construction itself (without a proof for the general case) can be tackled informally by students throughout the secondary school who are familiar with basic constructions (perpendicular line through a point, square given a side, etc.) and have basic knowledge of enlargements.

There are many similar problems involving constructions that can be investigated or even solved with different degrees of formality through an investigation with GeoGebra. Two such problems could be, for example, given a square, constructing a square with double area using only a straight edge, or given three parallel lines constructing an equilateral triangle with a vertex lying on each line.

A.2 Problem 2 – Calculus and Cubic Graphs

Statement of the problem

1. Describe the locus of the point of inflection of $f(x) = x^3 + 2x^2 - kx + 3$ for $k \in \mathbb{R}$.

2. Describe the locus of the point of inflection of $f(x) = kx^3 + 2x^2 - x + 3$ for $k \in \mathbb{R}$.

GeoGebra Investigation

The initial investigation of this problem relies on the use of a slider to define the constant k. To begin with, a slider should be created with no specific initial conditions (other than naming it k) as these can be changed at any stage. The function can then be sketched in terms of the value k defined by the slider and the point of inflection can be generated using the 'InflectioPoint' function. Alternatively, one could find the inflection by differentiating twice, finding the intersection of the second derivative with the x-axis and finding the point on function f with the same x-coordinate as the intersection. This second approach may be taken by students with less familiarity of Computer Algebra or who are more comfortable with a geometric approach to investigations of this kind.

Once the point of inflection is generated, its trace can be turned on so that when the value of k is changed using the slider, its locus is shown. In the case of part 1, the outcome should be a vertical line, while in the case of part 2, it should be a parabola, as shown in Figure A.2.1 below. Varying the range of k and the amount it's increased by can help verify this, however, unlike Problem 1, this does not provide a solution to the problem but allows the student to make a conjecture.

Algebraic Solution

1. In order to prove that the locus of the inflection of $f(x) = x^3 + 2x^2 - kx + 3$ for $k \in \mathbb{R}$ is a vertical line, it suffices to show that the x-coordinate of said point does not depend on the value of k. In order to do so, the coordinates of the inflection can be calculated using calculus as follows. $f'(x) = 3x^2 + 4x - k$ and $f''(x) = 6x + 4$. Therefore, the x-coordinate of the inflection is $x = -\frac{2}{3}$, which is independent of k.

2. A similar argument can be followed for the case where $f(x) = kx^3 + 2x^2 - x + 3$ and the coordinates of the point of inflection can be calculated in terms of k as follows. Firstly, note that if k = 0, then there is no inflection as the function would be quadratic in this case, so it can be assumed that $k \neq 0$.

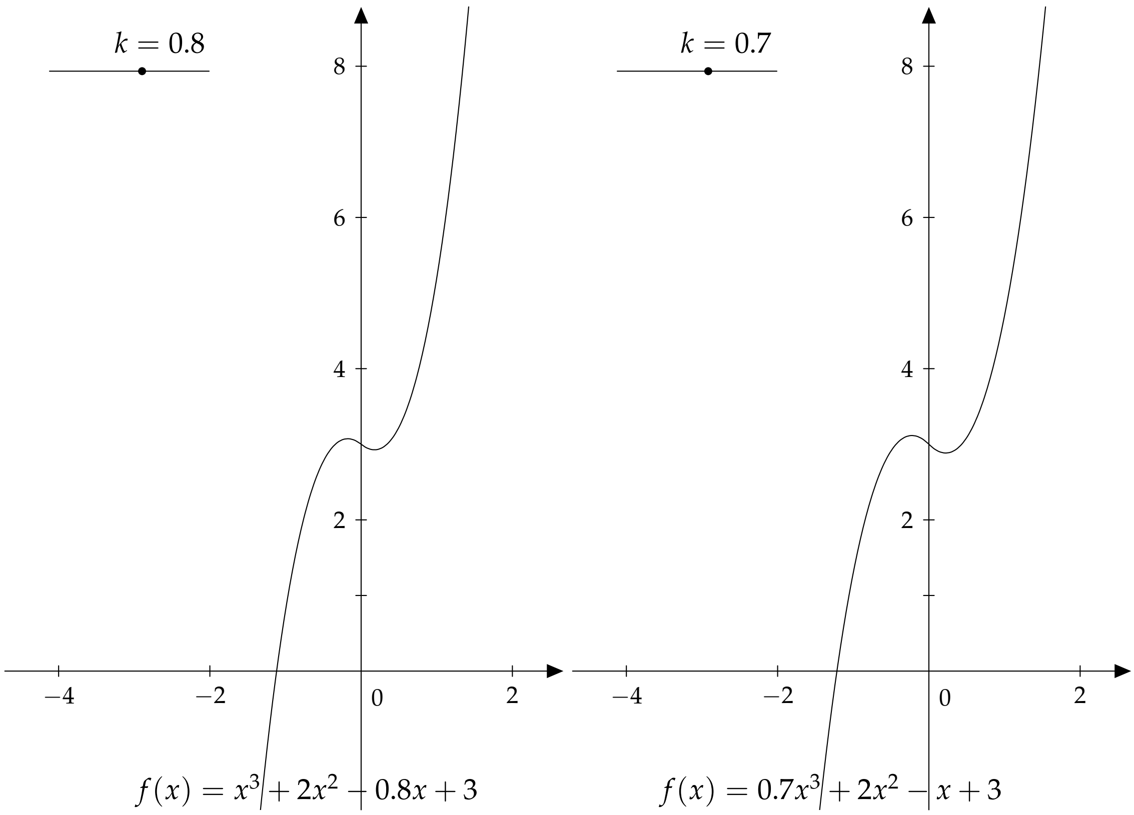

Figure A.2.1: Left: Problem 1 – $f(x) = x^3 + 2x^2 - kx + 3$. Right: Problem 2 – $f(x) = kx^3 + 2x^2 - x + 3$.

3. Differentiating twice it can be shown that $f''(x) = 6kx + 4$ and hence, the x-coordinate of the inflection is $x = -\frac{2}{3k}$. In order to study the behaviour of the inflection the substitution $a = -\frac{2}{3k}$ can be made and since $k \in \mathbb{R} \setminus \{0\},\ a \in \mathbb{R} \setminus \{0\}$. Therefore, $f(a) = ka^3 + 2a^2 - a + 3 = (-\frac{2}{3a})a^3 + 2a^2 - a + 3 = \frac{4}{3}a^2 - a + 3$ and the coordinate of the point of inflection is given by $(a, \frac{4}{3}a^2 - a + 3)$, which has a parabolic locus.

Interesting Points, Extensions and Similar Problems

It can be seen from the algebraic solution that the initial investigation is not required for the proof of the conjectures. However, without it it would be very difficult to make the required conjectures. It is important to note that there are several pieces of software other than GeoGebra that would allow this type of investigation, such as 'Autograph' (which relies on the constant controllers). The advantage of using GeoGebra rather that other software is that similar investigations can be carried out with more geometric ideas such as conic sections and their relation to loci, for example, describing an ellipse as the locus of a point such that the sum of its distances from two fixed points is constant, as shown in Figure A.2.2 below.

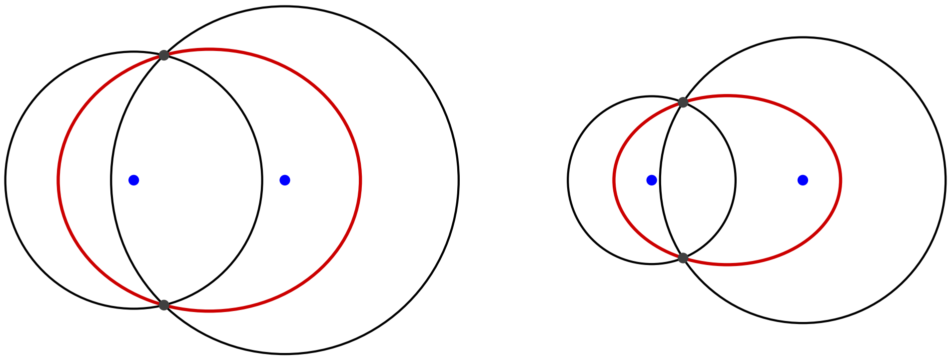

Figure A.2.2: On the left the constant sum is 3 and on the right the constant sum is 4. In both cases, the locus of the point clearly lies on the ellipse in red.

Furthermore, GeoGebra would allow the more general problem of describing the locus of the point of inflection of $f(x) = ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{R}$ and three of the constants are fixed while the fourth varies along $\mathbb{R}$. A simple investigation following the ideas presented above would suggest that if a varies, the locus is a parabola, if b varies, the locus is a cubic, and if c or d vary, the locus is a vertical line, all of which can be proven algebraically by calculating the coordinates of the point of inflection. Similar investigation can be carried out with other polynomial functions or by looking at the relation between the locus of the point of inflection and the roots of cubic functions.

Appendix B

Pölya's advice

First

UNDERSTANDING THE PROBLEM

You have to understand the problem

- What is the unknown? What are the data? What is the condition?
- Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory?
- Draw a figure. Introduce suitable notation.
- Separate the various parts of the condition. Can you write them down?

Second

DEVISING A PLAN

Find the connection between the data and the unknown. You may be obliged to consider auxiliary problems if an immediate connection cannot be found. You should obtain eventually a *plan* of the solution.

- Have you seen it before? Or have you seen the same problem in a slightly different form?
- *Do you know a related problem?* Do you know a theorem that could be useful?
- *Look at the unknown!* And try to think of a familiar problem having the same or a similar unknown.
- *Here is a problem related to yours and solved before. Could you use it?* Could you use its result? Could you use its method? Should you introduce some auxiliary element in order to make its use possible?
- Could you restate the problem? Could you restate it still differently? Go back to definitions.
- If you cannot solve the proposed problem try to solve first some related problem. Could you imagine a more accessible related problem? A more general problem? Am more special problem? An analogous problem? Could you solve a part of the problem? Keep only a part of the condition, drop the other part; how far is the unknown then determined, how can it vary? Could you derive something useful from the data? Could you think of other data appropriate to determine the unknown? Could you change the unknown or the data, or both if necessary, so that the new unknown and the new data are nearer to each other?
- Did you use all the data? Did you use the whole condition? Have you taken into account all essential notions involved in the problem?

Third

CARRYING OUT THE PLAN

Carry out your plan.

- Carrying out your plan of the solution *check each step*. Can you see clearly that the step is correct? Can you prove that it is correct?

Fourth

LOOKING BACK

Examine the solution obtained.

- Can you *check the result?* Can you check the argument?
- Can you derive the result differently? Can you see it at a glance?
- Can you use the result or the method, for some other problem?